Worldsheet Approach to Problems in String Cosmology

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Abstract

This thesis focuses on applying worldsheet conformal field theories to study problems of time-dependence and moduli stabilisation as motivated by string cosmology. Firstly, we construct exact string backgrounds using conformal field theories where the compact sectors are chosen such that the massless scalar moduli are naturally frozen. We term these models 'ϵ-Gepner models' as they are related to the usual algebraic compactifications called Gepner models. The sign of the charge deficit ϵ categorises the compactifications into two branches, one of whom requires D-branes to stabilise the dilaton potential, resulting in a small negative cosmological constant. Secondly, we discuss tools to study the timelike Liouville theory for the supersymmetric case. We attempt to define the timelike supersymmetric Liouville theory for the $\mathcal{N} = 1$ case via an analytic continuation of the shift relations of the spacelike theory. Our methods have not yet led to a diagonal two-point function for the timelike $\mathcal{N} = 1$ theory.

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Chapter 1

Introduction

The aim of this thesis is to explore string worldsheet techniques to study certain problems in string cosmology. We focus in particular on the problem of moduli stabilisation and that of understanding time-dependent string backgrounds.

Compactification of the ten dimensional target space of superstring leads to a vast choice for the data describing the compact Calabi-Yau manifold. Often, especially in supersymmetric compactifications, there are massless moduli fields that describe the deformations of the internal manifold. A fundamental problem in string phenomenology therefore is to find ways to construct vacua without any moduli since there are strong cosmological constraints on the existence of massless or very light scalar fields. A solution to this problem has been devised over the past few years in the form of flux compactifications. While they can preserve some supersymmetry, fluxes give vacuum expectation values to the massless fields and hence stabilise the moduli. According to the current wisdom, this leads to a large number of vacua known as the 'landscape' since fluxes can take many different discrete values.

Conformal field theoretic techniques developed for studying the string worldsheet are a rich source of tools to experiment with the precise sting theory backgrounds or vacua. We therefor attempt a worldsheet approach to address the problem. We construct exact backgrounds, which we term *ϵ-Gepner models* as they are related to the usual algebraic compactifications called 'Gepner models' originally studied by Gepner in 1987. These models perturb the usual charge split of $c = 6 + 9 = 15$ in the internal and external sectors of the worldsheet theory that correspond to the compact and the noncompact spaces respectively, by introducing a charge deficit ϵ such that $c = (6-\epsilon)+(9+\epsilon)$. The spectrum of these models naturally comes without moduli – the massless scalar field are a special feature of the usual Gepner models. Note that, as expected, these ϵ-Gepner models do not have a geometrical interpretation.

Motivated partially by studying time-dependent backgrounds and partially by the requirements of the external sector of our ϵ-Gepner models – that we require to be supersymmetric, we have attempted to better understand the timelike Liouville theory. Our approach has been to extend the methods devised by McElgin [72] to define the timelike bosonic Liouville theory to the supersymmetric case. The two-point function obtained by McElgin restricts the central charge to the values taken by $(1, q)$ minimal models. We wish to state from the outset that our attempts to define a timelike supersymmetric theory have largely led to negative results which we document in the thesis, with some of the more involved proofs left for the appendix. We should also note that the McElgin method places certain requirements on the two-point function that have since been argued unnecessary recently [79]. This opens further avenues for our future work in this domain.

The outline of the thesis is as follows: Chapters 2 and 3 review the string and conformal field theoretic ingredients we require in the remainder of the text and are based on standard literature. In Chapter 2, we motivate the themes explored in the thesis by describing the various problems in cosmology that are the subject of our study. This is a review section that sets the scene for the rest of the thesis. Beginning with preliminary string theory, we review some of the mainstream approaches aimed at solving the moduli stabilisation problem. Moreover, we motivate our work on the supersymmetric timelike Liouville theory via problems in string cosmology, namely inflation and time-dependence string backgrounds. Chapter 4 introduces the framework upon which our moduli stabilisation models are based. We introduce the charge deficit ϵ and motivate it via the effective field theory arguments. We discuss interpretational issues surrounding the $\epsilon > 0$ and $\epsilon < 0$ cases. Appendix B summarise the results of the computer-based searches for ϵ-Gepner models and their spectra. Together Chapters 4 and 7 and the Appendix B document the work published in the paper [63] discussing the closed string picture and the open string sector work that is yet to be published.

Chapter 5 discusses the noncompact sector of our worldsheet models. This sector is related to nonrational conformal field theories with continuous degrees of freedom. This is largely a review chapter with some exceptions such as the derivation of the mass formula for use in the spectrum of the ϵ-Gepner models and the computations pertaining to modular invariance of the $\mathcal{N} = 2$ Liouville theory. Chapter 6 contains our attempts to analytically continue the spacelike $\mathcal{N} = 1$ Liouville theory to the timelike one. Appendix A contains some of the detailed computations carried out by us in various aspects of the thesis.

Chapter 2

Some Problems in String Cosmology

String theory was first proposed as a theory of quantum gravity in 1974 [1]. Various discoveries during the 70's and the early 80's such as the perturbative finiteness of the superstring theories [2], the anomaly cancellation [3] and the heterotic string compactifications [4] rapidly led to establishing string theory as a viable candidate for a 'theory of everything' in the particle physics community. Most of these developments that occurred around mid-1980's were termed the 'first superstring revolution'. They still left some reasons to be wary of string theory's credentials since they were five competing versions of the theory. The 'second superstring revolution' in the mid-1990's led to a unified understanding of these varieties of string theories as different limits or aspects of one a single overarching theory – the M theory. The central idea in string theory is that of *duality* that linked for instance, the strong coupling limit of one theory to another weakly coupled theory. Gaps remain in further understanding to make connections with experiment, but these ideas have led to string theory or M theory providing a powerful framework for particle physics research.

Application of string theory to cosmology has seen some serious problems, in particular it has been difficult to see how testable predictions may be obtained from the theory. The cosmological constant problem remains unsolved. It would also be expected of a theory of quantum gravity to describe cosmological inflation. Many of these problems trace their roots back to a fundamental issue that sits at the heart of string theory: making choices of parameters such as the size and shape of the compactification manifold. These parameters are the massless scalar fields or *moduli*. Moreover, evolution of the universe we inhabit necessitates an understanding of time dependent solutions of string theory.

In this chapter, we aim to briefly describe two of the key problems in string cosmology that motivates this work, namely stabilisation of moduli and the study for time-dependent string backgrounds. We review some of the mainstream approaches that have been taken to solve these problems. These techniques are varied but are in general referred to as *flux compactifications* and are different from those we attempt to develop in this work in that we seek to apply the worldsheet conformal field theory techniques to moduli stabilisation and time dependence. We start with a brief overview of string theory fundamentals in the next section. Section 2.2 discusses basic setup of inflation in cosmology and describes what is meant by the cosmological constant problem. In sections 2.3, we discuss the moduli stabilisation problem.

2.1. String Backgrounds

2.1.1. Classical Dynamics of String

While the dynamical trajectory of a particle in Minkowski space is its worldline, a one-dimensional string sweeps out a two-dimensional *worldsheet.* Coordinates τ and σ are conventionally used to denote the timelike and spacelike coordinates of this worldsheet respectively. For closed strings whose ends are joined together, the spacelike coordinate is periodic $\sigma \in [0, 2\pi)$. The string worldsheet is an object that lives in the ambient spacetime, referred to as the *target space.* The embedding in a D-dimensional Minkowski spacetime is formulated by the map $X^\mu(\tau, \sigma)$, $\mu = 0, 1, \ldots, D-1$. For closed string, we impose $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$.

The simplest action that describes the dynamics of this string may be derived from the metric on a general metric $\gamma_{\alpha\beta}$ on the worldsheet. Since the worldsheet is embedded in the target spacetime, $\gamma_{\alpha\beta}$ is the induced metric, i.e., the pullback of the flat Minkowski metric on the target spacetime

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.1}$$

where $\partial_\alpha = \partial/\partial\sigma^\alpha$ for the worldsheet coordinates $\sigma^\alpha = \sigma, \tau$. The so-called *Nambu-Goto action* proportional to the area of the worldsheet may then be written

$$S_{\mathrm{NG}} = -T \int d^2\sigma \sqrt{-\det\ \gamma}. \tag{2.2}$$

Here T is a constant of proportionality, interpreted as the string tension. T is often written in terms of the Regge slope α' as $T = 1/(2\pi\alpha')$. α' is a length scale that sets the size of the string. The action (2.2) is not easy to quantise since it is highly nonlinear. An action that is equivalent to the Nambu-Goto action at the classical level, because it gives rise to the same equations of motion, is the string sigma model action or the *Polyakov action*, given by

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det g}\; g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.3}$$

The equation of motion for X^μ in this action is

$$\partial_\alpha(\sqrt{-\det g}\; g^{\alpha\beta}\partial_\beta X^\mu) = 0, \tag{2.4}$$

which coincides with that computed with (2.2). The field $g_{\alpha\beta}$ is a dynamical field given [14] by $g_{\alpha\beta} = 2f(\sigma)\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ for a conformal factor $f(\tau,\sigma)$. Replacing $g_{\alpha\beta}$ in the action (2.3) with its equation of motion $g_{\alpha\beta} = 2f\gamma_{\alpha\beta}$ retrieves the original action (2.2). The energy momentum tensor is defined to be

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-\det g}}\frac{\partial S}{\partial g^{\alpha\beta}}, \tag{2.5}$$

where we have chosen conventions of [14].

Both Nambu-Goto (2.2) and Polyakov (2.3) actions have Poincaré and diffeomorphism invariances. The Poincaré symmetry $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$ is a global symmetry of the action for rotations Λ and translations a^μ. The diffeomorphism symmetry is a local reparameterisation of coordinates $\sigma^\alpha = \tilde{\sigma}^\alpha(\tau,\sigma)$ which leaves the action invariant, but under which the fields X^μ transform as worldsheet scalars, while the field $g_{\alpha\beta}$ transforms as a tensor. The Polyakov action additionally exhibits *Weyl invariance* under which the metric transforms by a conformal factor, $g_{\alpha\beta}(\tau,\sigma) \to \Omega(\tau,\sigma)g_{\alpha\beta}(\tau,\sigma)$ where $\Omega(\tau,\sigma)$ is some positive function.

The Weyl symmetry allows us to 'choose a gauge' in which the equations of motion (2.4) may be greatly simplified. First we use the diffeomorphism invariance to make the metric locally conformally flat by setting

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \tag{2.6}$$

for some function $\phi(\tau, \sigma)$. This is known as the *conformal gauge.* We can further use the Weyl invariance on the sphere to set

$$g_{\alpha\beta} = \eta_{\alpha\beta}, \tag{2.7}$$

thus ending up with a flat Minkowski metric on the worldsheet. Choosing this gauge means that the Polyakov action (2.3) becomes a theory of D free scalar fields and the X^μ equation of motion reduces to the free wave equation

$$\partial_\alpha\partial^\alpha X^\mu = 0. \tag{2.8}$$

The energy momentum tensor $T_{\alpha\beta}$ may be computed to be

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X^\nu \eta_{\mu\nu}. \tag{2.9}$$

Since we have chosen the gauge (2.7), the equations of motion (2.8) are subject to the constraints $T_{\alpha\beta} = 0$.

In the worldsheet lightcone coordinates $\sigma^\pm = \tau \pm \sigma$, the equations of motion (2.8) read

$$\partial_+\partial_- X^\mu = 0, \tag{2.10}$$

which has the general solution

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^-) + X^\mu_R(\sigma^+) \tag{2.11}$$

for the left and right moving waves X^μ_L and X^μ_R respectively. For the periodicity conditions

$X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau)$ imposed on the closed string, the Fourier modes corresponding to the most general, periodic solution are given by

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+} \tag{2.12a}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}. \tag{2.12b}$$

The terms x^μ and p^μ have the interpretation of the centre of mass position and momentum of the string, while the coefficients obey $\tilde{\alpha}_n^\mu = (\tilde{\alpha}_n^\mu)^*$ and $\alpha_n^\mu = (\alpha_n^\mu)^*$ for X^μ to be real.

The constraints $T_{\alpha\beta} = 0$ in lightcone coordinates reduce to $(\partial_+ X)^2 = (\partial_- X)^2 = 0$. By defining $\alpha_0^\mu = \sqrt{\alpha'/2}\, p^\mu$ as the zero-th mode of the Fourier expansion in (2.12), we see that the constraint $(\partial_+ X)^2 = 0$ lead to

$$\alpha' \sum_n L_n e^{-in\sigma^-} = 0, \qquad \text{where} \qquad L_n = \frac{1}{2}\sum_m \alpha_{n-m}^\mu \alpha_m^\nu \eta_{\mu\nu}, \qquad n \in \mathbb{Z}. \tag{2.13}$$

There is an analogous expression for $(\partial_- X)^2 = 0$ in terms of left moving Fourier modes of the constraint $\tilde{L}_n$ with $\tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\, p^\mu$. These are the Fourier modes of the energy momentum tensor (2.9). Any classical solution (2.12) must obey further constraints $L_n = \tilde{L}_n = 0$. These are known as *level matching*. The mass-shell condition may be derived from the zero mode constraints arising from L_0 and $\bar{L}_0$ as

$$p_\mu p^\mu = \frac{4}{\alpha'}\sum_{n>0}\alpha_n^\mu \alpha_{-n}^\nu \eta_{\mu\nu} = \frac{4}{\alpha'}\sum_{n>0}\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu \eta_{\mu\nu} \tag{2.14}$$

This condition or more accurately its quantum version that we will see in the next section will play an important role in computing the spectrum of fields in our ϵ-Gepner models.

2.1.2. Quantisation

In the covariant quantisation of the string, the fields $X^\mu(\sigma,\tau)$ and their conjugate momenta

$$P_\mu(\sigma,\tau) = \frac{1}{2\pi\alpha'}\eta_{\mu\nu}X^\nu(\sigma,\tau) \tag{2.15}$$

are promoted to operator-valued fields $\hat{X}^\mu$ and $\hat{P}_\mu$ satisfying the usual equal-time commutation relations

$$[\hat{X}^\mu(\sigma,\tau), \hat{X}^\nu(\sigma',\tau)] = [\hat{P}_\mu(\sigma,\tau), \hat{P}_\nu(\sigma',\tau)] = 0, \tag{2.16a}$$

$$[\hat{X}^\mu(\sigma,\tau), \hat{P}_\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\delta^\mu_\nu, \tag{2.16b}$$

where $\delta(\sigma-\sigma')$ is the Dirac delta function and δ^μ_ν is the Kronecker's delta. Note that we will largely drop the hats from our operators explicitly from now on. From the commutation relations (2.16), we can compute the commutation relations between the Fourier modes x^μ, p^μ, α^μ_n and $\tilde{\alpha}^\mu_n$ of the expansion (2.12),

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \qquad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta_{\mu\nu}\delta_{n,-m}. \tag{2.17}$$

Operators α^μ_n and $\tilde{\alpha}^\mu_n$ act as creation operators for $n<0$ and annihilation operators for $n>0$. We can build the Fock space of the theory by first introducing a vacuum state $|0\rangle$ that obeys $\alpha^\mu_n|0\rangle = \tilde{\alpha}^\mu_n|0\rangle = 0$ for $n>0$ and then acting on $|0\rangle$ with the $n<0$ creation operators.

In order to impose the quantum mechanical version of the constraints $T_{\alpha\beta}$, we will now switch to lightcone coordinates $\sigma^\pm = \tau \pm \sigma$, in which the worldsheet metric takes the form $ds^2 = -d\sigma^+ d\sigma^-$ and the components of the energy momentum tensor are given by $T_{\pm\pm} = \partial_\pm X^\mu \partial_\pm X^\nu \eta_{\mu\nu}$ and $T_{\pm\mp} = 0$. In terms of the oscillators, these constraints may be written similarly to the classical theory (2.13) where L_n's will now be quantum mechanical operators. Note that products of these operators are normal-ordered products with the annihilation operators acting on the right of the creation operators. We will not denote such products explicitly here using the standard $:\cdot:$ notation. The physical states $|\text{phys}\rangle$ of our Hilbert space are those that are annihilated by positive frequency modes of the energy momentum tensor. This leads to the quantum mechanical version of the mass-shell condition (2.14) given by

$$L_n|\text{phys}\rangle = 0, \qquad (L_0 - a)|\text{phys}\rangle = 0 \tag{2.18}$$

and similarly for the right moving modes. The constant a is called the normal ordering constant that may be calculated by various methods such as BRST quantisation or the ζ function regularisation. We can see that the expectation value of the energy momentum tensor vanishes $\langle\text{phys}|L_n|\text{phys}\rangle = \langle\text{phys}|\bar{L}_n|\text{phys}\rangle = 0$ for all $n \neq 0$ since the state on the right in each expression is annihilated by the positive frequency modes, while by taking the Hermitian conjugate we can see that the state on the left is annihilated by the negative frequency modes.

Let us briefly look at the low lying states in the spectrum of open and closed strings. For open strings, the Neumann boundary conditions $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$ lead to identification of the oscillators, $\alpha_n^\mu = \tilde{\alpha}_n^\mu$. We will not review the spectrum in detail, only comment that rewriting the L_0 physical state condition in (2.18) in terms of the number operator N that gives the 'level' of creation and annihilation operators, we obtain

$$\Big(p_\mu p^\mu - \frac{4}{\alpha'}(N-1)\Big)|\text{phys}\rangle = 0, \tag{2.19}$$

where we have used the fact that $a = 1$ and the factor of $4/\alpha'$ arises as a result of the choice of normalisation in the definitions of the modes (2.12). The $N = 0$ states are tachyonic since $p^2 = -m^2$. At $N = 1$, there is a massless state $|A_\mu\rangle = A_\mu(p)\alpha_n^\mu|0\rangle$. The L_n constraint in (2.12) for $n > 0$ leads to a condition $\partial^\mu A_\mu = 0$ which is the Lorentz gauge condition for an electromagnetic potential.

For the closed string, the space coordinate is periodic, $\sigma \sim \sigma + 2\pi$ and we therefore have two independent sets of left and right moving oscillators. In addition to the condition (2.19), we will have an analogous version for the right moving sector with the number operator denoted by $\tilde{N}$. The $N = 0$ states are once again tachyonic. At $N = 1$, we find a massless state $|S_{\mu\nu}\rangle = S_{\mu\nu}(p)\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0\rangle$. $S_{\mu\nu}$ is a spacetime tensor which under the Lorentz group $\mathrm{SO}(1, D-1)$ decomposes into a symmetric traceless part $G_{\mu\nu}$, anti-symmetric part $B_{\mu\nu}$ and a trace part ϕ. These are called the metric, the Kalb-Ramond field and the dilaton respectively. The states at levels $N > 1$ will be massive.

2.1.3. Coupling to the Dilaton and Beta Functions

A generalisation of the Polyakov action (2.3) seen above is to a general curved spacetime,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det g}\; g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X), \tag{2.20}$$

with $g_{\alpha\beta}$ still the worldsheet metric as before. Note that the spacetime metric now is a function of the coordinates X^μ. The action (2.20) is known as a *non-linear sigma model.* By computing the quantum fluctuations around the flat space i.e., when the spacetime metric $G_{\mu\nu}(X)$ is a small perturbation of the flat Minkowski metric $\eta_{\mu\nu}$,

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.21}$$

we can see that the vertex operator associated with the perturbation field $h_{\mu\nu}$ in indeed that of the graviton.

Extending this further, the action of the string moving in the background with all three massless fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$ is given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,\Big[G_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu\epsilon^{\alpha\beta} + \alpha'\phi(X)R^{(2)}\Big]. \tag{2.22}$$

We can think of this worldsheet action as a two-dimensional quantum gravity coupled to some matter in form of some scalar fields.

The coupling to the dilaton vanishes for a flat worldsheet where $R^{(2)} = 0$. A vital consistency requirement for a string theory is invariance under a *Weyl transformation*, which is a local rescaling of the worldsheet metric

$$g_{\alpha\beta}(\tau,\sigma) \to e^{2\omega(\tau,\sigma)}g_{\alpha\beta}(\tau,\sigma), \tag{2.23}$$

for small $\omega(\tau,\sigma)$. For a constant dilaton

$$\phi(X) = \phi_0, \tag{2.24}$$

the term in the action inducing coupling to the dilaton

$$S_\phi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\;\phi(X)R^{(2)}, \tag{2.25}$$

reduces to

$$S_\phi = \phi_0 \chi, \tag{2.26}$$

where $\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\, R^{(2)}$ is the Euler character of the worldsheet. Such choice of a constant dilaton leaves the action Weyl invariant. This constant mode of the dilaton is normally taken to be the asymptotic value of the dilaton

$$\phi_0 = \lim_{X\to\infty} \phi(X), \tag{2.27}$$

and the string coupling constant is given by

$$g_s = e^{\phi_0}. \tag{2.28}$$

This implies that the for the string perturbation theory to be valid, the string must be localised to the regions of spacetime where $e^{\phi(X)} \ll 1$ for all X. When $e^{\phi(X)}$ is of order 1, we would need to employ non-perturbative techniques to study the theory. This places a strong requirement on the validity of perturbation theory.

A general dilaton $\phi(x)$ on the other hand, does break Weyl invariance since under (2.23), the Ricci scalar transforms as

$$R \to e^{-2\omega}(R - 2\nabla^2\omega). \tag{2.29}$$

The lack of Weyl classical invariance may be compensated by one-loop contributions arising from couplings to $G_{\mu\nu}$ and $B_{\mu\nu}$. The beta functions associated with $G_{\mu\nu}$, $B_{\mu\nu}$ and $\phi(X)$ at the one loop level are

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi + \frac{\alpha'}{4}H_{\mu\lambda\rho}H_\nu^{\lambda\rho} \tag{2.30}$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\phi H_{\lambda\mu\nu} \tag{2.31}$$

$$\beta(\phi) = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\phi + \alpha'\nabla^\mu\phi\nabla_\mu\phi - \frac{\alpha'}{24}H_{\lambda\mu\nu}H^{\lambda\mu\nu}. \tag{2.32}$$

The Weyl invariance requirement now becomes

$$\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\phi) = 0. \tag{2.33}$$

The string propagating in the curved spacetime may be thought of as propagating in a background configuration made up of its massless modes. The quantum conformal invariance leads us to mass shell conditions for these modes. Ignoring the higher order corrections, the beta function equations be seen as the equations of motion for the background fields $G_{\mu\nu}, B_{\mu\nu}$ and ϕ. We can work our way backwards and write down the action that these equations of motions can be derived from. This action is known as the *low-energy effective action* and is given by

$$S_{\text{eff}} = \frac{1}{\alpha'} \int d^{26}X \sqrt{-g} e^{-2\phi} \left(R + 4(\mathcal{D}\phi)^2 - \frac{1}{2 \cdot 3!} H^2 \right) + \ldots, \tag{2.34}$$

where the ellipsis denotes higher order terms in α' and derivatives. The coupling to a linear dilaton modifies the beta function equations and the effective action we have written down. This will be revisited when we start describing the ϵ-Gepner models in Chapter 4.

2.2. Inflationary Cosmology and The Cosmological Constant Problem

Before we discuss moduli stabilisation itself, we will review some issues in string cosmology, in particular the cosmological constant problem, which has been a focus of research in theoretical physics and cosmology in one way or another for the most of the last century. The string backgrounds we construct – the so-called ϵ-Gepner models – will provide a potential for the dilaton that will induce a small negative cosmological constant. This would correspond to an anti-de sitter (AdS) solution. Techniques exist to 'lift' this solution to a de Sitter (dS) vacuum with positive cosmological constant required by experiment. Details of these techniques are outside of the scope of this thesis, but we will briefly touch upon them in section 2.3.2. Application of string theory to cosmology requires understanding time dependence. This motivates our study of the time dependent Liouville theory.

On large scales, the universe is spatially homogeneous and isotropic. We hence look for solutions to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.35}$$

that describe the universe as a $\mathbb{R} \ltimes M$ spacetime where $\mathbb{R}$ denotes the time direction and M a homogeneous, isotropic and maximally-symmetric 3-manifold. This leads to the Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big], \tag{2.36}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the 2-sphere, k is the *curvature parameter* ($k = 1, 0, -1$ for positively curved, flat and negatively curved spatial sections) and $a(t)$ is the *scale factor*. To obtain a Robertson-Walker solution to Einstein's equations, we must satisfy the two Friedmann equations

$$H^2 \equiv \Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.37}$$

where ρ and p are energy density and pressure of a perfect fluid that models the energy-momentum tensor in an isotropic and homogeneous universe.

The observational data at the beginning of the twentieth century supported the idea that universe was non-expanding or static. Einstein, therefore, searched for static solutions to his equations i.e., solutions with $\dot{a} = 0$. He subsequently modified his equations to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.38}$$

with Λ a new free parameter, called the *cosmological constant*. As a result, the Friedmann equations become

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.39}$$

These equations indeed admit a static solution with $k > 0$ and $\rho, p, \Lambda \geq 0$.

A subsequent discovery made by Hubble that the universe indeed is expanding spelt trouble for a theory with cosmological constant, showing that there was no longer a need for static solution. Einstein abandoned the idea calling it his "biggest blunder". Unfortunately it turns out that, we cannot set $\Lambda = 0$ and move on; particle physics does not allow us that freedom [17, 18]. Quantum fluctuations in the vacuum of the standard model contribute to $\langle T_{\mu\nu} \rangle$ in a way that mimics cosmological constant. Moreover, the current observational data favours a small positive cosmological constant i.e., a de Sitter spacetime.

A positive cosmological constant accelerates the universal expansion, while a negative cosmological constant and/or ordinary matter tend to decelerate it [17]. The 'standard cosmology' suffers from a number of shortcomings. Inflation, at present, is seen as the most compelling framework to address some of the problems of the standard cosmology. The most prevalent approach today is called the *slow-roll inflation.* We can derive the conditions for slow-roll inflation by analysing the action of a real scalar field in a four dimensional curved spacetime with metric $g_{\mu\nu}$

$$S = \int d^4x \, \sqrt{-g} \, \{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \}. \tag{2.40}$$

Assuming ϕ to be spatially homogeneous i.e., $\partial_i \phi = 0, i = x, y, z$, its equation of motion in the Robertson-Walker metric is given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2.41}$$

where the dot denotes the time derivative and V' is the derivative with respect to ϕ. The Friedmann equation for this theory is

$$H^2 = \frac{1}{3M_P^2} \Big(-\frac{1}{2}\dot{\phi}^2 + V(\phi) \Big), \tag{2.42}$$

where M_P is the Planck mass. Inflation can occur if the evolution of ϕ sufficiently gradual such that the potential energy dominates its kinetic energy and the second derivative $\ddot{\phi}$ is small enough for this state of affairs to be maintained for a sufficiently long period of time [16]. We, therefore, need $\dot{\phi}^2 \ll V(\phi)$ and $|\ddot{\phi}|^2 \ll |3H\phi|^2, |V'|$. We can state these two conditions in terms of the *slow-roll parameters* ϵ, η as following

$$\epsilon \equiv \frac{1}{2}M_P^2\left(\frac{V'}{V}\right)^2 \ll 1, \qquad \eta \equiv M_P^2\left(\frac{V''}{V}\right) \ll 1. \tag{2.43}$$

The ϵ condition ensures that there is a period of accelerated expansion, while the η condition works to guarantee that this period lasts sufficiently long in order to address the problems that inflation intends to solve, namely the horizon and flatness problems. We will not describe these problems in detail here.

In string theory, it is known that at weak string coupling $g_s \to 0$ and/or at large volume $R \gg l_s$ for the compact dimensions (l_s being the string length scale), the slow-roll conditions (2.43) are not true [31]. All known sources of potential energy fall too rapidly to zero [27, 28, 29] to allow slow-roll inflation. Thus, to achieve slow-roll inflation we must work in the strong coupling or small radius regime, which at present is poorly understood [31]. We are therefore forced to work with rapidly-rolling fields or moduli that then have to be stabilised. Section 2.3 addresses this issue of freezing or stabilising the moduli. The KKLT construction described in section 2.3.2 is one such scenario.

2.3. Moduli Stabilisation

2.3.1. Moduli in String Theory

In a usual string background, the spacetime manifold $\mathfrak{M}$ is made up of a compact and a noncompact part

$$\mathfrak{M} = \mathfrak{M}_{\text{ext}} \times \mathfrak{M}_{\text{int}}. \tag{2.44}$$

The dimension of the manifold $\mathfrak{M}$ – the target space of string theory – is fixed by local scale invariance to be 26 for a bosonic theory and 10 for the supersymmetric case. These are referred to as the *critical dimensions* of string theory. To illustrate the mechanics of such a split of the target space and its physical consequences, we will take a simple example of a compactification for the bosonic string where the internal manifold is a circle S^1. That is one of the 26 spacetime dimensions is compactified while the remaining directions are flat Minkowski space,

$$\mathbb{R}^{1,24} \times S^1. \tag{2.45}$$

The circle S^1 has radius R and is parameterised by the coordinate X^{25}, i.e., $X^{25} = X^{25} + 2\pi R$. This general idea has been around since the 1920's and is known as *Kaluza-Klein compactification* [9, 10]. If we consider length scales much larger than R, then the spacetime fields may be thought of as independent of the coordinate X^{25}. The full $D = 26$ metric $G^{(26)}_{\mu\nu}$ in the Einstein frame can be decomposed into three fields that now depend only on $R^{1,24}$: $\tilde{G}^{(25)}_{\mu\nu}$, a vector A_μ and a scalar field ρ, with $\mu\nu = 0, 1, \ldots, 24$. The full metric of the $D = 26$ spacetime now becomes

$$ds^2 = \tilde{G}^{(25)}_{\mu\nu} dX^\mu dX^\nu + e^{2\rho}(dX^{25} + A_\mu dX^\mu)^2. \tag{2.46}$$

The Ricci scalar in 26 dimensions $\mathcal{R}^{(26)}$ is given by

$$\mathcal{R}^{(26)} = \mathcal{R}^{(25)} - 2e^{-\rho}\nabla e^{\rho} - \frac{1}{4}e^{2\rho}F_{\mu\nu}F^{\mu\nu}. \tag{2.47}$$

Inserting (2.46) and (2.47) into the 26-dimensional Einstein-Hilbert action

$$\frac{2}{\kappa^2}\int d^{26}X\sqrt{-\det G^{(26)}}\,\mathcal{R}^{(26)},$$

we obtain

$$\frac{2\pi R}{\kappa^2}\int d^{25}X\sqrt{-\det G^{(25)}}\,\Big(\mathcal{R}^{(25)} + \partial_\mu\rho\partial^\mu\rho - \frac{1}{4}e^{2\rho}F_{\mu\nu}F^{\mu\nu}\Big). \tag{2.48}$$

We see no mass term for the scalar ρ in this action. This is how Kaluza and Klein were able to show that the reduction of D dimensional theory of gravity to $D-1$ dimensions resulted in $D-1$ dimensional gravity coupled to a $U(1)$ gauge theory induced by the vector field A_μ and a single massless scalar. There is no potential for the field ρ governing its vacuum expectation value.

Changing the value of the scalar field corresponds to changing the radius R of the compactified direction X^{25}. All compactifications of the Kaluza-Klein type suffer from this problem that there are massless scalar fields that parameterise the compactified directions. Massless scalar fields such as the dilaton and the field ρ are called the *moduli*. In a generic compactification of the string, the metric in the compact dimensions depends on such continuous parameters in a manner described here. No particular values of these parameters appear to be preferred [23]. Such moduli have not been observed in nature. Therefore, for a string compactification to successfully model the universe we live in, the values of the moduli must be fixed. This is referred to as *moduli stabilisation*.

2.3.2. Flux Compactifications and KKLT

Flux compactifications are the most prevalent technique for moduli stabilisation. Moduli fields are stabilised dynamically in flux compactifications by introducing non-vanishing antisymmetric tensor fields, the fluxes, in the compact space. An n-form potential A with and $(n+1)$-form field strength $F = dA$, induces a magnetic flux of the form $\int_{\gamma_{n+1}} F$, which only depends on the homology of the cycle γ_{n+1}. The dual field strength in D dimensions induces an electric flux in a similar way. Many good reviews of flux compactifications exist; see [23] and [31] for some examples. We will not discuss the generic setup in detail here. Instead we will comment briefly on a particular model known as KKLT, based on the seminal paper [30] that builds on techniques introduced in [25]. Note that although, we use the KKLT mechanism to partially set the scenario for problems we study in the thesis, reviewing its technical details is outside the scope of this work. The remainder of this section lists the salient points of the KKLT approach; for detailed arguments we refer the reader directly to their work [30].

Late-time or late-universe cosmology is concerned with large length scales and low-energies. It provides a way to probe physics on the largest observable length scales in the universe [16]. Motivated by constructing models for late-time cosmology where there is a small positive cosmological constant, there has been a great deal of interest in the quantum nature of de Sitter (dS) spacetime, more specifically in finding its ground states or vacua. This is also inherently difficult since there can be no supersymmetry in a de Sitter spacetime [19]. In [30], Kachru, Kallosh, Linde and Trivedi (KKLT) propose a scenario for constructing metastable dS vacua, in which they first construct a supersymmetric Anti-de Sitter (AdS) vacuum with all moduli stabilised, followed by breaking supersymmetry in a way that leads to a dS solution.

For a split of the type (2.44), KKLT start with a warped IIB string compactification i.e., a geometry where the scale factor affecting the four dimensional noncompact manifold (in superstring theory) depends on the coordinates of the six dimensional compact Calabi-Yau manifold

$$ds^2 = w(y)^{-1} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + g^{(6)}_{mn} dy^m dy^n. \tag{2.49}$$

In a supersymmetric theory not required to be renormalisable, the effective potential is completely determined by a superpotential W, which is a holomorphic function of the chiral superfields and a Kähler potential K, which is a function of both chiral and antichiral superfields [15]. In the KKLT construction, background fluxes are turned on that generate a superpotential for the compact Calabi-Yau manifold

$$W_0 = \int_{\mathfrak{M}_{\text{int}}} G_3 \wedge \Omega, \tag{2.50}$$

where G_3 is a linear combination of Neveu-Schwarz (NS) and Ramond-Ramond (RR) fluxes and Ω is the holomorphic 3-form on $\mathfrak{M}_{\text{int}}$. W_0 is the tree level contribution. The full superpotential is given by $W(\rho) = W_0 + Ae^{ia\rho}$, where ρ is a single Kähler volume modulus and the coefficient A is determined by the energy scale below which the supersymmetric QCD is valid. The exponential term is the correction that comes from nonperturbative effects [30, 13] from Euclidean D3-brane instantons or gaugino condensation from wrapped D7-branes. The choice of the source determines the factor a in the exponential. These corrections help stabilise the volume modulus ρ of the Kähler potential $K = -3\ln(-i(\rho - \bar{\rho}))$, where contributions from other moduli are left out as they are fixed by hand.

Kähler moduli control the size of the compact manifold and KKLT wish the Kähler potential to be of 'no-scale type' in order for the η-condition of inflation (2.43) to be satisfied [35], which is to say that in $\mathcal{N} = 1$ supergravity, where the F-term scalar potential is given by

$$V = e^K \left(\sum_{a,b} g^{a\bar{b}} D_a W \, \overline{D_b W} - 3|W|^2 \right), \tag{2.51}$$

with a, b running over all the moduli, the sum over Kähler moduli cancels the term $-3|W|^2$, leaving us with a 'no-scale' potential

$$V_{\text{no-scale}} = e^K \sum_{i,j} g^{i\bar{j}} D_i W \, \overline{D_j W}. \tag{2.52}$$

where i, j label all fields exlcuding ρ. With the potentials W and K given above, the scalar potential V can be extremised by solving the supersymmetric condition on W that $D_\rho W = \partial_\rho W + W \partial_\rho K = 0$, leading to a supersymmetric minimum, given by

$$V_{\text{AdS}} = -3e^K |W|^2. \tag{2.53}$$

This is an AdS vacuum since it has negative cosmological constant. KKLT claim that adding antibranes ($\overline{D3}$ branes) in a way that can be found in their paper [30] produces a nonsupersymmetric dS vacuum from a supersymmetric AdS vacuum. They further claim that this solution is metastable and has a lifetime greater than the cosmological timescale of 10^{10} years and hence should produce conditions conducive to inflation. Later in the thesis we will describe how the dilaton potential can be stabilised by introducing D-branes in our ϵ-Gepner models. This procedure leads to an anti de Sitter vacuum as is the case for KKLT i.e., a solution with negative cosmological constant is obtained. KKLT further study this solution and 'lift' it to a de Sitter vacuum. This lifting of AdS to dS will not be discussed in our ϵ-Gepner models.

2.3.3. Other approaches

Other approaches that share the basic premise of flux compactification with KKLT include the large volume compactification of [34]. We will not discuss them here; see [35] for a good review. Among numerous non-worldsheet approaches to moduli stabilisation the models of [22] and [62] are noteworthy. Becker *et al* [22] are motivated by the need for considering non-geometric backgrounds since for any geometric compactification, the internal space will have at least one free parameter or a modulus, namely the overall size of the manifold or the Kähler modulus. They stabilise the complex structure moduli and the dilaton by turning on appropriate RR and NSNS fluxes. They find that type IIB orientifolds provide examples of constructions with intrinsically no Kähler moduli and can result in four-dimensional theories that describe Minkowski as well as anti-de Sitter vacua. They share some CFT ingredients with our constructions, but we will not discuss them in this thesis. Antoniadis *et al* [62] compute an effective action for D-branes in type I string theory using nonlinear supersymmetry, which gives rise to new contributions to the scalar potential. Minimising this potential leads to moduli-stabilised vacua including an anti-de Sitter vacuum for a non-critical dilaton and a de Sitter solution when the criticality is preserved.

Chapter 3

Conformal Field Theory

The study of the two dimensional string worldsheet is a rich subject in its own right. It goes under the name of conformal field theory, which is a quantum field theory that is covariant under conformal transformations. In two dimensions, the case most relevant to us in string theory, there is an infinite-dimensional group of local conformal transformations, described by holomorphic functions. Conformal field theory has many applications outside of string theory, particularly in statistical mechanics. Although in recent years, conformal field theories in higher dimensions have gained a lot of attention due to their role in the AdS/CFT correspondence, they will not be covered in this thesis for they are not of direct concern to us.

3.1. Preliminary Review

Beginning from the rudimentary building blocks in d dimensions, our aim in this section will be to describe the conformal field on the complex plane. Much of the formulation traces its roots back to the seminal paper of Belavin, Polyakov and Zamalodchikov, henceforth referred to as the BPZ formulation [58].

3.1.1. Conformal symmetry in d dimensions

We start by a formulation of the conformal symmetry that underpins the conformal field theory. Consider a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ of signature (p, q) on a flat d dimensional Minkowski spacetime $\mathbb{R}^{p,q}$ with $d = p + q$. The line element or the infinitesimal interval between two points in the space endowed with this metric is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Under a change of coordinates, $x \mapsto x'$, the $(0,2)$ metric tensor transforms as

$$g_{\mu\nu} \mapsto g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x). \tag{3.1}$$

The conformal group consists of a subset of these transformations called *conformal transformations*, which leave the metric invariant up to a scale

$$g'_{\mu\nu}(x') \mapsto g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \tag{3.2}$$

for some positive function $\Omega(x)$. These are consequently the transformations that preserve the angle $v \cdot w/\sqrt{v^2 w^2}$ between two vectors $v, w \in \mathbb{R}^{p,q}$, where $v \cdot w = g_{\mu\nu}v^\mu w^\nu$. The semidirect product of translations and Lorentz transformations of flat space, namely the Poincaré group, is a subgroup of the conformal group, since taking $\Omega(x) = 1$ leaves the metric invariant: $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$.

Under the infinitesimal form of these transformations $x^\mu \to x^\mu + \epsilon^\mu$, the metric in the flat space $g_{\mu\nu} = \eta_{\mu\nu}$ transforms as follows

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu. \tag{3.3}$$

where $\partial_\mu = \partial/\partial x^\mu$. This must be proportional to the metric in order to satisfy (3.2)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}, \tag{3.4}$$

where the constant of proportionality is determined by taking a trace of the equation by $\eta_{\mu\nu}$. By rearranging, we obtain

$$\big(\eta_{\mu\nu}\partial_\lambda \partial^\lambda + (d-2)\partial_\mu \partial_\nu\big)(\partial \cdot \epsilon) = 0. \tag{3.5}$$

For $d = 2$, this equation simplifies and we shall return to this case shortly since it will be of main concern to us in string theory. For $d > 2$, we find that ϵ may be at most quadratic in x in order for the third derivative of ϵ to vanish (required by equations (3.4) and (3.5)). The constant case, where $\epsilon^\mu = a^\mu$, is that of ordinary *translations* by vectors $a^\mu \in \mathbb{R}^{p,q}$. The cases when ϵ is linear in x are *rotations* $\epsilon^\mu = \omega_\nu{}^\mu x^\nu$ with ω an antisymmetric matrix and *scale transformations* or *dilatations* $\epsilon^\mu = \lambda x^\mu$. Finally, the quadratic choice may be stated as $\epsilon^\mu = b^\mu x^2 - 2x^\mu(b\cdot x)$ and is termed *special conformal transformations*. Locally, the generators $a^\mu\partial_\mu$, $\omega^\mu{}_\nu\epsilon^\nu\partial_\mu$, $\lambda x\cdot\epsilon$ and $b^\mu(x^2\partial_\mu - 2x^\mu x\cdot\partial)$ form an algebra isomorphic to $SO(p+1,q+1)$ [43].

The condition of conformal invariance places constraints on the n-point functions of the quantum theory. In general, the theory would satisfy certain properties:

a) There is a set of fields $\{A_i\}$ containing in general an infinite number of fields and their derivatives.

b) There is a set of *quasi-primary* fields $\{\phi_j\}\subset\{A_i\}$ which transform under the global conformal transformations belonging to $O(p+1,q+1)$ as follows

$$\phi_j(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d} \phi_j(x'), \tag{3.6}$$

where Δ_j is the *conformal dimension* of the field ϕ_j. The n-point functions (vacuum expectation values of products of fields) are covariant under this transformation

$$\langle\phi_1(x_1)\dots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \times\ \langle\phi_1(x_1')\dots\phi_n(x_n')\rangle \tag{3.7}$$

c) All other fields in the set $\{A_i\}$ may be expressed as linear combinations of the quasi-primary fields and their derivatives.

d) There is a vacuum state $|0\rangle$ that is invariant under the global conformal transformations.

3.1.2. Specialising to $d = 2$ and the Virasoro Algebra

In three or more dimensions, the conformal group is finite dimensional. In two dimensions, however, there exist an infinite variety of coordinate transformations that are locally conformal: they are the holomorphic mappings from the complex plane to itself.

Let us start by considering coordinates t and x on a two-dimensional flat space equipped with a metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1)$. This space will be referred to as the *worldsheet* with reference to string theory. In the $d = 2$ Euclidean metric where $g_{\mu\nu} = \delta_{\mu\nu}$, the condition (3.5) becomes

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1, \tag{3.8}$$

where $\partial_1 = \partial_t$ and $\partial_2 = \partial_x$. These coincide with the Cauchy-Riemann conditions on a function to be analytic. In other words, we can write the transformation parameter ϵ as a function of the complexified coordinate $w = t + ix$ and its complex conjugate $\bar{w} = t - ix$ and find that the conformal transformations in two dimensions amount to analytic coordinate transformations of the type $w \mapsto f(w)$ and $\bar{w} \mapsto \bar{f}(\bar{w})$ where $f(w)$ and $\bar{f}(\bar{w})$ are holomorphic and anti-holomorphic respectively.

In physics, we usually start work with a compact space i.e., a cylinder as spacetime to avoid spacetime infrared divergences. So, we will switch to another flat worldsheet, the cylinder $\Sigma = \mathbb{R} \times S^1$ with coordinates (t, x). Changing to Euclidean lightcone coordinates, the cylinder may be mapped to the complex plane via

$$z = e^{t+ix} \quad \text{and} \quad \bar{z} = e^{t-ix}, \tag{3.9}$$

with the line element given by

$$ds^2 = dz d\bar{z} \tag{3.10}$$

In these new complex coordinates, the conformal invariance means that all transformations

$$z \mapsto \xi(z) \quad \text{and} \quad \bar{z} \mapsto \bar{\xi}(\bar{z}), \tag{3.11}$$

are conformal transformations of the complex plane. The line element (3.10) is now given by [43]

$$ds^2 = \left|\frac{\partial \xi}{\partial z}\right|^2 dz d\bar{z}. \tag{3.12}$$

Hence, we have $\Omega(z) = |\partial\xi/\partial z|^2$. Transformations ξ and $\bar{\xi}$ form two copies of the group of diffeomorphism, $\mathrm{Diff}(S^1)$. This contains as subgroup the group of globally well-defined transformations given by

$$\xi(z) \mapsto \frac{az+b}{cz+d} \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \tag{3.13}$$

with an analogous copy for $\bar{z}$. This group is called the Möbius group $\mathrm{PSL}(2,\mathbb{R})$. All other transformations in $\mathrm{Diff}(S^1)$ are local transformations only: they are invertible only locally in the neighbourhood z.

The chiral splitting (3.11) that naturally occurs in two dimensions means that we can treat z and $\bar{z}$ independently. Quantities that depend on z and $\bar{z}$ will be referred to as *left* and *right moving* respectively and can be treated independently as they do not interact with each other as far as the symmetries properties are concerned. The terms relate to the lightcones in the original cylindrical coordinates.

For the infinitesimal transformations, we take the basis

$$z \to z' = z + \epsilon_n(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}_n(\bar{z}), \qquad (n \in \mathbb{Z}) \tag{3.14}$$

where

$$\epsilon_n(z) = -z^{n+1} \quad \text{and} \quad \bar{\epsilon}_z(\bar{z}) = -\bar{z}^{n+1}. \tag{3.15}$$

The corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \qquad (n \in \mathbb{Z}). \tag{3.16}$$

These span what is known as the *Witt algebra* whose commutation relations are given by

$$[l_n, l_m] = (n-m)l_{n+m} \tag{3.17}$$

and similarly for $\bar{l}$ with $[l_n, \bar{l}_m] = 0$. The Witt algebra is the Lie algebra of $\mathrm{Diff}_+(S^1)$, the orientation-preserving diffeomorphisms of the circle S^1.

Symmetry generators in general can be constructed via the application of the Noether theorem [43]. In classical field theory, the Noether current for general coordinate transformation is the energy-momentum tensor $T_{\mu\nu}$, which is symmetric. $T_{\mu\nu}$ is conserved and in conformally invariant theories, also traceless $T_\mu{}^\mu = 0$.

In the complex coordinates (3.9), the components of the metric are $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$. Since, $T_{\mu\nu}$ is traceless, $T_{z\bar{z}} = T_{\bar{z}z} = 0$. Secondly, since $T_{\mu\nu}$ is conserved, $g^{\lambda\mu}\partial_\lambda T_{\mu\nu} = 0$, we have

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0, \tag{3.18}$$

leading to

$$\partial_{\bar{z}} T_{zz} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} = 0. \tag{3.19}$$

Therefore, the only non-vanishing components of $T_{\mu\nu}$ in complex coordinates are

$$T(z) = T_{zz}(z) \quad \text{and} \quad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}). \tag{3.20}$$

Thus energy-momentum tensor splits into left and right movers: $T(z,\bar{z}) = T(z) + \bar{T}(\bar{z})$. The generators (3.16) now become the Laurent modes of $T(z)$ and $\bar{T}(\bar{z})$.

These remaining components of the energy-momentum tensor generate local conformal transformations on the complex z-plane. In radial quantisation, the conserved charge is hence given by [43]

$$Q = \frac{1}{2\pi i} \oint \big\{ dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \big\}, \tag{3.21}$$

where the line integral is performed over some circle of fixed radius such that both the dz and $d\bar{z}$ integrations are taken in the counter-clockwise direction.

There is a central extension of the Witt algebra called the *Virasoro algebra*. The commutator relations of the Virasoro algebra generators L_n, $n \in \mathbb{Z}$ are,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{3.22}$$

where c is the *central charge* such that $[c, L_n] = 0, \forall n$. This is the left-moving Virasoro algebra, and because of the chiral splitting, we have a right-moving copy with generator $\bar{L}_n$ obeying a similar relation with a central charge of $\bar{c}$. We shall assume the central charges of these two copies to be the same. Every CFT carries a representation of this algebra with some value of c and $\bar{c}$.

Using Noether theorem, we can calculate the energy-momentum tensor for the Virasoro algebra generators and write the left-moving (holomorphic) and right-moving (anti-holomorphic) components (3.20) in terms of the Virasoro generators L_n and $\bar{L}_n$,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n. \tag{3.23}$$

A generalisation of the transformation law (3.12) allows us to write a transformation for the field $\Phi(z, \bar{z})$ as

$$\Phi(z,\bar{z}) \to \left(\frac{\partial\xi}{\partial z}\right)^h \left(\frac{\partial\bar{\xi}}{\partial\bar{z}}\right)^{\bar{h}} \Phi(\xi(z), \bar{\xi}(\bar{z})), \tag{3.24}$$

where h and $\bar{h}$ are real-valued and $\bar{h}$ is not h's complex conjugate. A field $\Phi(z, \bar{z})$ that transforms in this manner is called a *primary field* of *conformal weight* $(h, \bar{h})$. The fields that do not transform in this way are called *secondary*. A primary field is always quasi-primary, but secondary fields may not necessarily be so.

The variation of $\Phi(w, \bar{w})$ is given by its equal-time commutator with the charge Q (3.21)

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) &= [Q, \Phi(w,\bar{w})] \\ &= \frac{1}{2\pi i}\oint\Big\{[dzT(z)\epsilon(z), \Phi(w,\bar{w})] + [d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \Phi(w,\bar{w})]\Big\}.\end{aligned} \tag{3.25}$$

Since these product are only defined for $|z| > |w|$ in radial quantisation, we define the *radial-*

ordering operator product ρ of two operators $A(z)$ and $B(w)$ as

$$\rho(A(z), B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases}. \tag{3.26}$$

This allows us to write the equal-time commutator $[A(z), B(w)]$ as a contour integral around the point w. Equation (3.25) may now be written as

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) &= \frac{1}{2\pi i}\Big(\oint_{|z|>|w|} - \oint_{|z|<|w|}\Big)\Big\{dz\epsilon(z)\rho\big(T(z)\Phi(w,\bar{w})\big) + d\bar{z}\bar{\epsilon}(\bar{z})\rho\big(\bar{T}(\bar{z})\Phi(w,\bar{w})\big)\Big\} \\ &= \frac{1}{2\pi i}\oint\Big\{dz\epsilon(z)\rho\big(T(z)\Phi(w,\bar{w})\big) + d\bar{z}\bar{\epsilon}(\bar{z})\rho\big(\bar{T}(\bar{z})\Phi(w,\bar{w})\big)\Big\} \\ &= h\partial\epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial\Phi(w,\bar{w}) + \bar{h}\bar{\partial}\bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\bar{\partial}\Phi(w,\bar{w}), \end{aligned} \tag{3.27}$$

where $\partial = \partial/\partial w$ and $\bar{\partial} = \partial/\partial\bar{w}$ and we have used (3.24) to write down the end result. In order that the charge (3.21) induce the correct infinitesimal conformal transformations (3.14), the radial-ordering operator product ρ of generators T and $\bar{T}$ with fields $\Phi(w,\bar{w})$ must satisfy

$$\begin{aligned} \rho\big(T(z),\Phi(w,\bar{w})\big) &= \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \ldots \\ \rho\big(\bar{T}(\bar{z}),\Phi(w,\bar{w})\big) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \ldots. \end{aligned} \tag{3.28}$$

This is defined to be the so-called *operator product expansions* (OPE) of T and $\bar{T}$ with $\Phi(w,\bar{w})$. In the rest of the text, ρ will be dropped. In general, the singularities that occur when two operators $A(x)$ and $B(y)$ approach one another are encoded in their operator product expansion

$$A(x)B(y) \sim \sum_i C_i(x-y)O_i(y), \tag{3.29}$$

where $O_i(y)$ are all local operators and $C_i(x-y)$ are singular numerical coefficients that can be determined via dimensional analysis. In 2-dimensional conformal field theories, we can consider a basis of operators Φ_i with fixed conformal weights h_i. If for two such operators Φ_i

and Φ_j, we make the following choice of normalisation for their two-point function:

$$\langle \Phi(z,\bar{z})\Phi(w,\bar{w})\rangle = \delta_{ij}\frac{1}{(z-w)^{2h_i}}\frac{1}{(\bar{z}-\bar{w})^{2\bar{h}_i}}, \tag{3.30}$$

then the operator product expansion will be given as follows

$$\Phi_i(z,\bar{z})\Phi_j(w,\bar{w}) \sim \sum_k C_{ijk}\,(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\;\Phi_k(w,\bar{w}), \tag{3.31}$$

where the C_{ijk} are constants. We will now consider some examples of operator product expansions, starting with a free boson. Note however, that the OPE of the energy-momentum tensor (3.23) with itself is always given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots \tag{3.32}$$

where c is the central charge of the theory as captured by the algebra-defining commutation relations (3.22).

Examples: As an exercise we will explicitly compute the OPEs and central charges of a free boson and a free fermion. For a single free real boson φ, the energy-momentum tensor $T(z)$ is given by

$$T(z) = -\frac{1}{2}:\partial\varphi(z)\partial\varphi(z):. \tag{3.33}$$

The OPE of $T(z)$ with itself is given by

$$T(z)T(w) = \frac{1}{4}:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w):. \tag{3.34}$$

Using Wick's theorem of quantum field theory, we can write

$$T(z)T(w) = \frac{1}{4}\Big\{2\langle\partial\varphi(z)\partial\varphi(w)\rangle\langle\partial\varphi(z)\partial\varphi(w)\rangle + 4\langle\partial\varphi(z)\partial\varphi(w)\rangle :\partial\varphi(z)\partial\varphi(w):\Big\}, \tag{3.35}$$

where the factors of 2 and 4 arise as a result of counting all possible combinations of terms.

Feynman propagators of bosonic fields φ and its derivatives are given by

$$
\begin{aligned}
&\langle \varphi(z)\varphi(w)\rangle = -\log(z-w) + \dots, \qquad \langle \partial\varphi(z)\varphi(w)\rangle = -\frac{1}{z-w} + \dots \\
&\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2} + \dots,
\end{aligned}
\tag{3.36}
$$

which allow us to write the OPE as

$$
T(z)T(w) = \frac{1}{2}\frac{1}{(z-w)^4} - \frac{1}{(z-w)^2}\, {:}\partial\varphi(z)\partial\varphi(w){:}\, . \tag{3.37}
$$

The z-dependent field in the normal-ordered term ${:}\,\partial\varphi(z)\partial\varphi(w)\,{:}$ on the right hand side can be expanded into its Taylor series around w. Then

$$
\begin{aligned}
{:}\partial\varphi(z)\partial\varphi(w){:} &= {:}\big\{\partial\varphi(w) + (z-w)\partial^2\varphi(w) + \mathcal{O}((z-w)^2)\big\}\partial\varphi(w){:} \\
&= {:}\partial\varphi(w)\partial\varphi(w){:} + (z-w)\, {:}\partial^2\varphi(w)\partial\varphi(w){:} + \mathcal{O}((z-w)^2) \\
&= -2T(w) - (z-w)\partial_w T(w) + \mathcal{O}((z-w)^2),
\end{aligned}
\tag{3.38}
$$

since $\partial_w T(w) = -\frac{1}{2}\partial_w\big\{{:}\,\partial\varphi(w)\partial\varphi(w)\,{:}\big\} = -\frac{1}{2}\big\{2\ {:}\,\partial^2\varphi(w)\partial\varphi(w)\,{:}\big\} = -\ {:}\,\partial^2\varphi(w)\partial\varphi(w)\,{:}$. Substituting (3.38) into (3.37), we get

$$
T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \tag{3.39}
$$

where we have ignored the $\mathcal{O}((z-w)^2)$ terms since they are non-singular after multiplying with $1/(z-w)^2$ and hence are not included in the operator product expansion. Using (3.32), we can read off the central charge $c = 1$ for a boson.

Similarly, for a theory with a single free real fermion Ψ with energy-momentum tensor

$$
T(z) = -\frac{1}{2}\, {:}\Psi(z)\partial\Psi(z){:} \tag{3.40}
$$

and Feynman propagators of the fermionic fields given by

$$
\begin{aligned}
\langle\Psi(z)\Psi(w)\rangle &= \frac{1}{z-w}+\dots, & \langle\partial\Psi(z)\Psi(w)\rangle &= -\frac{1}{(z-w)^2}+\dots \\
\langle\Psi(z)\partial\Psi(w)\rangle &= \frac{1}{(z-w)^2}+\dots, & \langle\partial\Psi(z)\partial\Psi(w)\rangle &= -\frac{2}{(z-w)^3}+\dots,
\end{aligned}
\tag{3.41}
$$

we can calculate the $T(z)T(w)$ OPE

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{4}\Big\{\frac{1}{(z-w)^4}+\frac{8T(w)}{(z-w)^2} \\
&\quad +\frac{2}{z-w}\Big(:\partial\Psi(w)\partial\Psi(w):+2\partial_w T(w)-:\partial\Psi(w)\partial\Psi(w):\Big)\Big\} \\
&= \frac{1/4}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{z-w}.
\end{aligned}
\tag{3.42}
$$

Now we can see that $c=1/2$ for a fermion.

Highest Weight Representations of the Virasoro Algebra: The state space in the theory are a direct sum $\mathcal{H}=\oplus_{(i,\imath)\in I}\mathcal{H}_i\otimes\mathcal{H}_\imath$ where I is some index set and Hilbert spaces $\mathcal{H}_i$ and $\mathcal{H}_\imath$ represent the left and right moving sectors respectively. We demand that the energy momentum tensor is bounded from below and the spaces $\mathcal{H}_i$ and $\mathcal{H}_\imath$ are highest weight representations, i.e., they contain highest weight states that satisfy

$$
L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0, n>0. \tag{3.43}
$$

In a CFT, one always demands that the vacuum state, i.e., a highest weight state with $h=0$ is in the state space. Given a highest weight state $|h\rangle$, all other states of the form $L_{-n_1}\dots L_{-k_k}|h\rangle$ for $n_i>0$ are known as descendent states.

We can introduce *conformal characters* $\chi_i(q)$ to count the degeneracy of the energy levels in a representation $\mathcal{H}_i$,

$$
\chi_i = \mathrm{Tr}_{\mathcal{H}_i} q^{L_0-c/24} \tag{3.44}
$$

where q is a formal parameter.

3.1.3. The Superconformal Algebra

The Virasoro algebra of the previous section is a bosonic algebra. It may be extended to an algebra of a superconformal field theory (SCFT) by adding fermionic generators. The $\mathcal{N} = 1$ superconformal algebra may be obtained by including an additional generator $G(z)$ of conformal weight $3/2$, which is the worldsheet superpartner of the energy-momentum tensor $T(z)$ (3.23). As before, there is a right moving $\bar{G}(\bar{z})$ which is the superpartner of $\bar{T}(\bar{z})$. The operator product of $T(z)$ is given as in (3.32) and the following two operator products complete the algebra [41],

$$T(z)G(z) = \frac{3/2}{(z-w)^2}G(w) + \frac{\partial_w G(w)}{z-w} + \ldots, \tag{3.45a}$$

$$G(z)G(w) = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} + \ldots. \tag{3.45b}$$

Next we extend it to the $\mathcal{N} = 2$ superconformal algebra by adding another supercurrent. Now we have $G_1(z)$ and $G_2(z)$ both with conformal weight $3/2$ each. The operator product $T(z)T(w)$ is still given by (3.32) whereas $T(z)G_1(w)$ and $T(z)G_2(w)$ are both defined by (3.45a). Moreover, (3.45b) gives the operator products of $G_1(z)G_1(w)$ and $G_2(z)G_2(w)$. Introducing $G^{\pm}(z) = \frac{1}{\sqrt{2}}\big(G_1(z) \pm iG_2(z)\big)$, the operator products of the full set of algebra generators are now given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots, \tag{3.46a}$$

$$T(z)G^{\pm}(z) = \frac{3/2}{(z-w)^2}G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{z-w} + \ldots, \tag{3.46b}$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \ldots, \tag{3.46c}$$

$$G^{+}(z)G^{-}(w) = \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} + \ldots, \tag{3.46d}$$

$$J(z)G^{\pm}(w) = \pm\frac{G^{\pm}(w)}{z-w} + \ldots, \tag{3.46e}$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + \ldots. \tag{3.46f}$$

$J(z)$ has a conformal weight 1 and is the $U(1)$ current. In the manner of (3.23), we can write the new generators J and $G^{\pm}$ of the $\mathcal{N}=2$ superconformal algebra in terms of their modes,

$$J(z) = \sum_n z^{-n-1} J_n, \qquad G^{\pm}(z) = \sum_n z^{-(n\pm a)-\frac{3}{2}} G^{\pm}_{n\pm a}, \tag{3.47}$$

where the parameter a lies in the range $[0,1)$. Values of $a \in \mathbb{Z}$ correspond to anti-periodic boundary conditions, known as the Ramond boundary conditions, whereas the periodic boundary conditions with $a \in \mathbb{Z} + \frac{1}{2}$ are called the Neveu-Schwarz boundary conditions. In terms of the mode expansions (3.23) and (3.47), commutation relations of $\mathcal{N}=2$ algebra may be written as

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{3.48a}$$

$$[J_n, J_m] = \frac{c}{3} n \delta_{n,-m}, \tag{3.48b}$$

$$[L_n, J_m] = -mJ_{m+n}, \tag{3.48c}$$

$$[L_n, G^{\pm}_{m\pm a}] = \Big(\frac{n}{2} - (m \pm a)\Big) G^{\pm}_{n+m\pm a}, \tag{3.48d}$$

$$[J_n, G^{\pm}_{m\pm a}] = \pm G^{\pm}_{n+m\pm a}, \tag{3.48e}$$

$$\{G^{+}_{n+a}, G^{-}_{m-a}\} = 2L_{n+m} + (n-m+2a)J_{n+m} + \frac{c}{3}\Big((n+a)^2 - \frac{1}{4}\Big)\delta_{n,-m}. \tag{3.48f}$$

We can study the highest weight representations of this algebra in analogy to the bosonic case. In addition to the modes L_n of $T(z)$, we now have J_m and $G^{\pm}_r$ corresponding to generators J and $G^{\pm}$ respectively. Once again, we will assume the modes with positive indices to be annihilation operators and require that the highest weight states $|\phi\rangle$ be annihilated by their action. That is,

$$L_n|\phi\rangle = J_m|\phi\rangle = G^{\pm}_r|\phi\rangle = 0, \tag{3.49}$$

for $n, m, r > 0$. These modes satisfy the reality conditions,

$$(L_n)^{\dagger} = L_{-n}, \quad (J_m)^{\dagger} = J_{-m}, \quad (G^{\pm}_r)^{\dagger} = G^{\mp}_{-r}. \tag{3.50}$$

We also need to label the states with the eigenvalues of zero modes of these operators. In the Neveu-Schwarz sector of the theory with modes of the kind $G^{\pm}_{n\pm\frac{1}{2}}$, the only zero index modes are L_0 and J_0 (there is no $G^{\pm}_0$). Let their eigenvalues when acting on a state $|\phi\rangle$, be h_ϕ and q_ϕ such that

$$L_0|\phi\rangle = h_\phi|\phi\rangle, \quad J_0|\phi\rangle = q_\phi|\phi\rangle. \tag{3.51}$$

Then the representation can be built by the action of the creation operators L_{-n}, J_{-m}, and $G^{\pm}_{-r}$ $(n, m, r > 0)$ on the highest weight state $|\phi\rangle$.

In the Ramond sector where modes have integer indices, we have two additional zero modes: $G^{\pm}_0$ with eigenvalues $g^{\pm}_\phi$ when acting on a highest weight state $|\phi\rangle$. A Ramond ground state is a state $|\phi\rangle$ which is annihilated by both G^+_0 as well as G^-_0. Due to (3.48f), Ramond ground states satisfy $h = \frac{c}{24}$.

A superconformal primary field $\phi(z)$ acts on the vacuum to create a state $|\phi\rangle$ i.e., $|\phi\rangle = \phi(0)|0\rangle$. The field $\phi(z)$ has the following operator product expansions with the algebra generators:

$$T(z)\phi(w) = \frac{h_\phi}{(z-w)^2}\phi(w) + \frac{\partial_w\phi(w)}{z-w} + \ldots, \tag{3.52}$$

$$J(z)\phi(w) = \frac{q_\phi}{z-w}\phi(w) + \ldots, \tag{3.53}$$

$$G^{\pm}(z)\phi(w) = \frac{(G^{\pm}_{-1/2}\phi)(w)}{z-w} + \cdots = \frac{\tilde{\phi}^{\pm}(w)}{z-w} + \ldots, \tag{3.54}$$

where we are focusing on the z-dependence of ϕ. The field $\tilde{\phi}^{\pm}(z) = \left(G^{\pm}_{-1/2}\phi\right)(z)$ is known as the 'superpartner' of $\phi(z)$. A *chiral primary field* is associated with a state $|\phi\rangle$ which is annihilated by the operator $G^+_{-1/2}$, that is $G^+_{-1/2}|\phi\rangle = 0$. Thus the OPE of a chiral primary field with G^+ has no singularities, since from (3.54)

$$G^+(z)\phi(w) = \frac{(G^+_{-1/2}\phi)(w)}{z-w} + \text{reg} = \text{reg}. \tag{3.55}$$

Similarly, an *antichiral primary field* creates a highest weight state which is annihilated by $G^-_{-1/2}$. Combining the left and right moving parts, states may be of four types (c,c), (a,c), (a,a) and (c,a), where ‘c’ and ‘a’ denote chiral and antichiral states respectively.

From (3.48f), we can compute

$$\{G^-_{1/2}, G^+_{-1/2}\} = 2L_0 - J_0. \tag{3.56}$$

The expectation value of this commutator gives

$$\langle\phi|\{G^-_{1/2}, G^+_{-1/2}\}|\phi\rangle = \langle\phi|(2L_0 - J_0)|\phi\rangle = 2h_\phi - q_\phi \tag{3.57}$$

Since $|\phi\rangle$ is a chiral primary state, this must vanish and hence,

$$h_\phi = \frac{q_\phi}{2}. \tag{3.58}$$

Since $(G^+_{1/2})^\dagger = G^-_{1/2}$, the expectation value (3.57) is always non-negative in a unitary theory, i.e.

$$2h_\phi - q_\phi \geq 0 \quad \Rightarrow \quad h_\phi \geq \frac{q_\phi}{2}, \tag{3.59}$$

with equality occurring only when $|\phi\rangle$ is a chiral primary state.

From the definition (3.29), the operator product expansion of two chiral primary fields $\phi(z)\chi(w)$ is given by,

$$\phi(z)\chi(w) = \sum_i \mathbb{C}_{\phi\chi\psi_i}\,(z-w)^{h_{\psi_i}-h_\phi-h_\chi}\psi_i(w), \tag{3.60}$$

where ψ_i are fields of conformal weights h_{ψ_i}. Since $U(1)$ charges add upon operator products, and we know that $h_\phi = q_\phi/2$ for a chiral primary field ϕ (3.58), we have

$$q_\psi = q_\phi + q_\chi \quad \Rightarrow \quad h_\psi \geq h_\phi + h_\chi, \tag{3.61}$$

with the equality,

$$h_\psi = h_\phi + h_\chi \tag{3.62}$$

holding for chiral primary fields due to (3.58). It implies that there are no singular terms in the operator product of two chiral primary fields. For $z \to w$ in (3.60), the only surviving terms in the sum on the right hand side are the ones with chiral primary ψ_i. Thus, the chiral primary fields yield a non-singular and closed ring known as a *chiral primary ring* under the operation of operator product. Similarly for an antichiral primary field ψ, the following relation holds

$$h_\psi = -\frac{q_\psi}{2}. \tag{3.63}$$

Here we have assumed ϕ to be a holomorphic field. By considering all chiral and antichiral combinations results in four such rings, (c,c), (a,c), (a,a) and (c,a) with the last two being the conjugates of the first two.

3.2. Rational and Noncompact Theories

A rational conformal field theory is one whose Hilbert space consists only of a finite number of representations. These were originally introduced in [8]. Noncompact theories, also known as irrational theories, on the other hand have continuous representations. We are largely motivated to review the rational theories as they make the internal sector of our ϵ-Gepner models. The noncompact theories on the other hand are inserted in the external sector of ϵ-Gepner models. For a particular noncompact theory, namely Liouville field theory, we are also interested in its analytic continuation motivated by studying time-dependent string backgrounds. The noncompact theories will be further looked at in chapters 5 and 6.

3.2.1. Minimal Models

For a bosonic CFT, it is well known that the unitarity conditions for the highest weight representations of the Virasoro algebra lead to two possibilities for the central charge c and the conformal dimension h. Either

$$c \geq 1, \quad h \geq 0, \tag{3.64}$$

or if $c < 1$

$$c = 1 - \frac{6}{k(k+1)}, \quad h_{r,s}(k) = \frac{\big((k+1)r - ks\big)^2 - 1}{4k(k+1)}, \tag{3.65}$$

where the integers r, s are the labels of primary fields in the theory and they obey

$$1 \leq r \leq m-1, \quad 1 \leq s \leq r, \quad k \in \mathbb{Z}, k \geq 3. \tag{3.66}$$

Theories from the series with $c < 1$ are called *minimal models* of the Virasoro algebra. They are the simplest examples of *rational CFTs* which contain only a finite set of primary fields. Having a finite number of primary fields greatly simplifies the analysis of a conformal field theory and minimal models are often referred to as being ‘exactly soluble’.

For an $\mathcal{N} = 1$ minimal model (i.e. rational CFTs of the $\mathcal{N} = (1)$ superconformal algebra), the central charge is given by

$$c = \frac{3}{2}\left(1 - \frac{8}{(k+2)(k+4)}\right), \tag{3.67}$$

and the fields are labelled by two integers (r, s)

$$1 \leq r \leq k-1 \text{ and } 1 \leq s \leq k+1. \tag{3.68}$$

The transformation

$$(r, s) \longrightarrow (k-r, k+2-s) \tag{3.69}$$

leads to the same field. The conformal dimension of a field $\phi_{r,s}$ is given by

$$h_{r,s} = \frac{\big((k+2)r - ks\big)^2 - 4}{8k(k+2)} + \frac{1}{32}\big(1 - (-1)^{r-s}\big), \qquad k \geq 3. \tag{3.70}$$

The difference of labels $(r-s)$ determines the Neveu-Schwarz (NS) and Ramond (R) subsectors

$$\phi_{r,s} \in \begin{cases} \text{NS-sector}, & r-s \in 2\mathbb{Z} \\ \text{R-sector} & r-s \in 2\mathbb{Z}+1. \end{cases} \tag{3.71}$$

In the NS sector, to each superconformal primary field $\phi_{r,s}$, there is a superdescendant field $\tilde{\phi}_{r,s}$, which is primary with respect to the bosonic subalgebra. Generically, it has conformal weight $\tilde{h}_{r,s} = h_{r,s} + 1/2$ (the exception being the vacuum whose superdescendant is the supercurrent at conformal weight $3/2$).

The $\mathcal{N} = 2$ minimal models are coset theories given by

$$\frac{SU(2)_k \times U(1)_4}{U(1)_{2k+4}}, \tag{3.72}$$

where the subscripts denote the levels of the affine Lie algebras and the central charge is

$$c(k) = \frac{3k}{k+2}, \qquad k = 1, 2, \dots \ . \tag{3.73}$$

The irreducible representations (of the bosonic subalgebra) of this theory can be labeled by (l, m, s), where l refers to the $SU(2)_k$, s to the $U(1)_4$ in the numerator, and m to the $U(1)_{2k+4}$ in the denominator. These labels satisfy

$$\begin{aligned} &l = 0, 1, \dots, k, \qquad m = -k-1, -k, \dots, k+2 \qquad s = -1, 0, 1, 2, \\ &\text{with} \quad l + m + s \ \text{ even.} \end{aligned} \tag{3.74}$$

Representations with $s = 0, 2$ belong to the NS sector, while those with $s = \pm 1$ to the R-sector. Triples (l, m, s) and $(k - l, m + k + 2, s + 2)$ give rise to the same representation and are identified. The conformal dimension h and U(1)-charge q of the highest weight state with labels (l, m, s) are given by

$$h^l_{m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8}, \qquad q^l_{m,s} = \frac{m}{k+2} - \frac{s}{2} \tag{3.75a}$$

as as long as the labels (l,m,s) lie in the so-called *standard range*, i.e. they satisfy

$$l=0,1,\ldots,k, \quad |m-s|\le l, \quad s=-1,0,1,2, \quad l+m+s \text{ even}, \tag{3.76a}$$

or

$$l=1,2,\ldots,k, \qquad m=-l, \qquad s=-2. \tag{3.76b}$$

Every triple (l,m,s) may be brought into the standard range using the transformations

$$(l,m,s)\mapsto(l,m+2k+4,s), \qquad (l,m,s)\mapsto(l,m,s+4). \tag{3.77}$$

Representations can be grouped into pairs (l,m,s) and $(l,m,s+2)$, each such pair makes up a full $\mathcal{N}=2$ super Virasoro module; all states in the same bosonic sub-representation have the same fermion number (or U(1)-charge) modulo two.

3.2.2. Noncompact Theories

Noncompact theories have continuous set of representations. We will not cover their general theory in detail, but simply discuss the examples in Chapter 5.

3.3. Gepner Models and Exact worldsheet backgrounds

In [38], Gepner introduced algebraic string compactifications using exact CFTs instead of sigma-models on compact Calabi-Yau manifolds for the internal sector of the worldsheet theory. Gepner's internal CFTs are tensor products of minimal models of the $\mathcal{N}=2$ super Virasoro algebra, such that their central charges add up to $15-3D/2$ where D is the dimension of the non-compact spacetime. The full theory is then a (GSO-projected) tensor product with an external CFT.

The string backgrounds discussed here, with non-vanishing central charge deficit ϵ, are formed in close parallel to Gepner models, therefore we review the main building blocks of Gepner's construction.

A Gepner model is formed from a tensor product of r minimal models with central charges $c(k_i)$ together with D external free bosons and fermions. The latter make up an $\mathrm{SO}(d)_1$ current algebra, where $d = D - 2$ (working in the light cone gauge). In Gepner's original construction, the levels k_i are chosen such that

$$c_{\text{int}} \equiv \sum_{i=1}^{r} c(k_i) = 15 - \frac{3}{2}D \ , \tag{3.78}$$

For compactifications to $D = 4$ dimensions, we must have $3 < r \leq 9$ because $1 \leq c(k_i) \leq 3$; the most famous example is $(3, 3, 3, 3, 3)$ which (like many other Gepner models) can be related to a sigma model on a Calabi-Yau manifold, namely the quintic hypersurface in $\mathbb{C}P^4$. In the rest of the text, we will occasionally abbreviate the models by collecting the n repeated levels together as k^n. For instance, $(k_1^3, k_2) \equiv (k_1, k_1, k_1, k_2) = (k_1, k_2, k_1, k_1) = \ldots$. The quintic therefore may be written as (3^5). We will also use $c(k_1, k_2, \ldots, k_r)$ to refer to the sum $\sum_{i=1}^{r} c(k_i)$.

To ensure spacetime supersymmetry and tachyon-freedom, the tensor product of minimal models and external CFT needs to be subject to certain projections. Most importantly, one demands that the total fermion number of any state be odd (the GSO projection). We can write this in the form

$$q_{\text{ext}} + q_{\text{int}} \in 2\mathbb{Z} + 1 \tag{3.79}$$

with the sum of the minimal model charges

$$q_{\text{int}} = \sum_{j=1}^{r} \frac{m_j}{k_j + 2} - \frac{s_j}{2} \tag{3.80}$$

and the charge of the external fermions

$$q_{\text{ext}} = \frac{d}{2}\frac{s_0}{2} \ ; \tag{3.81}$$

here, $s_0 = -1, 0, 1, 2$ labels the irreducible representations (c, o, s, v) of $\mathrm{SO}(d)_1$, such that q_{ext} simply amounts to the external fermion number. States in the tensor product that do not satisfy (3.79) are projected out.

Apart from this GSO-projection, one also has to ensure that only states in tensor products of $r+1$ NS or R sectors are packaged together; this is enforced by removing all states which do not satisfy

$$\frac{d}{2}\frac{s_0}{2} + \frac{s_j}{2}. \in \mathbb{Z}\ . \tag{3.82}$$

Projecting out states from a CFT normally spoils modular invariance of the closed string partition function – unless one treats the projection as part of an orbifold procedure (more precisely, in the case of Gepner's construction, a simple current orbifolding). Modular invariance is then restored by adding suitable twisted sectors to the states left from the tensor product theory one started with. We refer to [38] for more details and for explicit expressions of the full partition function.

3.4. Boundary Conformal Field Theory

The conformal field theory discussed thus far in this chapter is the bulk theory that describes the tree level closed string theory. Tree level open string theory involves boundary conformal field theory on the upper half plane (or on the strip). We first start with a brief review of the formulation of the theory on the upper half plane before describing boundary states. This section is based on various reviews in literature; see the introductory sections of [46] for instance. Boundary states in the context of Gepner models seen in the previous section will be required by our analysis of the open string sector of the ϵ-Gepner models in section 7.2, where we will review their construction based on [46].

We use the complex coordinates $z = e^{t+i\sigma}$ introduced in the above case of the bulk theory, with t and σ thought of as time and space variables. The upper half plane is defined by Im $z \geq 0$. In the interior of the upper half plane, Im $z > 0$, the theory is the usual bulk CFT defined on the full plane, in particular the bulk fields have the usual OPE in the interior of the half plane. We will demand that no energy flows across the real line. This amounts to imposing the following condition on the holomorphic and anti-holomorphic components (3.20) of the energy momentum tensor

$$T(z) = \bar{T}(\bar{z}) \text{ for } z = \bar{z} \text{ i.e., Im } z = 0. \tag{3.83}$$

This allows us to construct generators $L_n^{(\mathbb{H})}$ of a single Virasoro algebra

$$L_n^{(\mathbb{H})} = \frac{1}{2\pi i}\int_C z^{n+1}T(z)dz - \frac{1}{2\pi i}\int_C \bar{z}^{n+1}\bar{T}(\bar{z})d\bar{z}, \tag{3.84}$$

where C is a semicircle in the upper half plane that end of the real line. The space of states $\mathcal{H}$ of the boundary theory then decomposes into a sum of irreducible Verma modules $\mathcal{H} = \oplus \mathcal{H}_i$. Primary fields in the boundary theory obey

$$[L_n^{(\mathbb{H})}, \phi(z,\bar{z})] = z^n(z\partial + h(n+1))\phi(z,\bar{z}) + \bar{z}^n(\bar{z}\bar{\partial} + \bar{h}(n+1))\phi(z,\bar{z}), \tag{3.85}$$

where $h, \bar{h}$ are the conformal dimensions of the field $\phi(z,\bar{z})$. We can generalise the theory to cases where there are other chiral fields $W(z)$ and $\bar{W}(\bar{z})$ of half-integer conformal dimension in addition to $L_n^{(\mathbb{H})}$. Assuming $W(z) = \bar{W}(\bar{z})$ will guarantee the action of an extended chiral algebra on the state space. There is no interpretation of this condition in terms of flow of energy across the real line as in the case of $T(z)$ and $\bar{T}(\bar{z})$. Hence, we will generalise this condition to

$$W(z) = \Omega(\bar{W})(\bar{z}) \text{ for } z = \bar{z}, \tag{3.86}$$

where Ω is a local automorphism acting on the space of chiral fields. As before for the $L_n^{(\mathbb{H})}$ case, we can construct modes $W_n^{(\mathbb{H})}$ of these new fields

$$W_n^{(\mathbb{H})} = \frac{1}{2\pi i}\int_C z^{n+h_W-1}W(z)dz - \frac{1}{2\pi i}\int_C \bar{z}^{n+h_W+1}\Omega(\bar{W})(\bar{z})d\bar{z}. \tag{3.87}$$

The spectrum of open strings (i.e. of boundary fields) can most conveniently be computed after having introduced so-called boundary states. These are (non-normalisable) linear combinations of states from the Hilbert space of the bulk CFT, which encode the boundary conditions imposed along the real line into the bulk CFT. The boundary state of the theory is a state $|\alpha\rangle$ in the state space of the full bulk theory such that the n-point correlators of the theory on the half plane can be rewritten in term sof the correlators on the full plane, sandwiched between a boundary state inserted at $|\xi| = 1$ and at $|\xi| = e^{\frac{2\pi^2}{\beta_0}}$,

$$\langle \Theta\alpha | e^{-2\pi^2 H/\beta_0} \phi_1^{(P)}(\xi_1, \bar{\xi}_1) \ldots \phi_n^{(P)}(\xi_n, \bar{\xi}_n) |\alpha\rangle \tag{3.88}$$

where $\phi_i^{(P)}(\xi_i, \bar{\xi}_i)$ are bulk on the complex plane with coordinates $\xi, \bar{\xi}$; the latter are related to the upper half plane coordinates z via $\xi = e^{2\pi i \ln z/\beta_0}$ and $\bar{\xi} = e^{-2\pi i \ln \bar{z}/\beta_0}$. H is the bulk Hamiltonian $H = L_0 + \bar{L}_0 - c/12$ and Θ the bulk CPT operator, see [46] for detail of the full setup. Using (3.86), one can see that $|\alpha\rangle$ obeys the following *gluing* or *Ishibashi conditions*

$$(L_n - \bar{L}_{-n})|\alpha\rangle = 0, \qquad (W_n - (-1)^{h_W} \Omega(\bar{W}_{-n}))|\alpha\rangle = 0. \tag{3.89}$$

Note that L_n and W_n are the modes of the bulk generators. These relations depend on the automorphism Ω, which is also referred to as 'gluing automorphism'. The gluing conditions may be solved to find the so-called *Ishibashi states* named after Ishibashi who showed in [48] that to each irreducible highest weight representation i of the algebra $\mathcal{A}$ generated by W_n on the space $\mathcal{H}_i$, we can associate a state $|i\rangle\rangle$ – the Ishibashi state– which is unique up to an overall constant such that $(W_n - (-1)^{h_W} \bar{W}_{-n}|i\rangle\rangle = 0$ for the trivial automorphism $\Omega = 1$. We take the algebra of right and left moving generators to be same $\mathcal{A}_R = \mathcal{A}_L = \mathcal{A}$. If $|i, N\rangle, N \in \mathbb{Z}_+$ is an orthonormal basis for $\mathcal{H}_i$, then

$$|i\rangle\rangle = \sum_{N=0}^{\infty} |i, N\rangle \otimes U|i, N\rangle, \tag{3.90}$$

where U is the anti-unitary operator on the total right moving, or chiral Hilbert space $\mathcal{H}_R = \oplus_i \mathcal{H}_i$ satisfying $U\bar{W}_n = (-a)^{h_W} \bar{W}_{-n} U$, i.e. U acts like a chiral CPT operator.

For a more general automorphism, there is an analogous Ishibashi state $|i\rangle\rangle_\Omega$ which implements the Ω-twisted gluing conditions,

$$(W_n - (-1)^{h_W} \Omega(\bar{W}_{-n}))|i\rangle\rangle_\Omega = 0. \tag{3.91}$$

These Ishibashi states twisted by Ω may be associated to any representation i of $\mathcal{A}$. In summary, to each automorphism Ω with a set of gluing conditions (3.91) and to each irreducible highest weight representation i of $\mathcal{A}$ on $\mathcal{H}_i$, there is a unique Ishibashi state $|i\rangle\rangle_\Omega$ which implements the gluing conditions. A general boundary state constructed as a linear combination of Ishibashi states

$$|\alpha\rangle\rangle_\Omega = \sum B^i_\alpha |i\rangle\rangle_\Omega \tag{3.92}$$

where B^i_α are some coefficients, will describe a boundary CFT with symmetry algebra $\mathcal{A}$.The coefficients B^i_α are constrained [45] by the requirement that (3.88) without bulk field insertions should give the open string partition function, i.e. provide a count of open string vertex operators supported by the brane described by the boundary state $|\alpha\rangle\rangle$. The partition function is given by [45]

$$Z_{\alpha\alpha}(q) = \langle\alpha|\tilde{q}^{L_0-c/24}|\alpha\rangle \tag{3.93}$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$ and τ is related to β_0 via $\beta_0 = -2\pi i\tau$; see [46].

The boundary states in our ϵ-Gepner models are constructed analogously to those in Gepner models which were first studied by [46]. We will not go into the details of these here; instead, we will present a brief review in section 7.2 and list the formulae we need to compute spectrum in concrete models.

Chapter 4

Effective Spacetime Physics with Charge Deficit

In chapter 2, we reviewed the spacetime techniques for moduli stabilisation, where the leading approach was flux compactifications. The models considered by flux compactification have been largely studied using effective field theory and supergravity. This requires that the compactification scale is large compared to the string scale but on the other hand is small compared to experimentally accessible scales. In this respect, such an approach seems somewhat contrived. Therefore it is of interest use a worldsheet approach which allows for an exact description at all length scales, even if the internal CFTs used in these compactifications do not necessarily have a geometrical limit.

As stated in section 2.1, the critical string theory carries the central charge of 15 for the full superstring background. The usual flat compactification comprises of a split of $6 + 9$ for central charges of CFTs corresponding to external and internal sectors respectively. The central charge of six for the external theory corresponds to four bosonic and four fermionic degrees of freedom. The models we consider here indeed provide critical string theory backgrounds – the overall charge is always 15. The central idea for us will be to introduce a *central charge deficit* to parameterise the central charges of both the internal and external CFTs and play with the $6 + 9$ split. The various ways of constructing models will be the central theme of this work.

4.1. The Charge Deficit ϵ

Consider the sigma-model action of the string moving in a background with massless fields, the graviton $g_{\mu\nu}$, the Kalb-Ramond field $b_{\mu\nu}$ and the dilaton ϕ,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\,\Big\{\gamma^{\alpha\beta}g_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu + \epsilon^{\alpha\beta}b_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu + \alpha'\mathcal{R}\phi(X)\Big\}, \tag{4.1}$$

where the $\alpha, \beta = 0, 1$ and $\mu, \nu = 0, 1, \ldots, D-1$ are the worldsheet and spacetime indices respectively. $\mathcal{R}$ is the Ricci scalar of the worldsheet metric $\gamma_{\alpha\beta}$. The string perturbation theory is valid wherever the string coupling constant $g_s = e^\phi$ is small. Although this action is classically conformally invariant, coupling to a generic non-constant dilaton as seen in 2.1.3, breaks the invariance in the quantum theory. To one-loop order one finds the conditions for conformal invariance are $\beta^g_{\mu\nu} = \beta^b_{\mu\nu} = 0$ where

$$\beta^g_{\mu\nu} = \alpha'\Big(R_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\mathcal{D}_\mu\mathcal{D}_\nu\phi\Big) + \mathcal{O}(\alpha'^2) \tag{4.2a}$$

$$\beta^b_{\mu\nu} = \alpha'(\mathcal{D}^\lambda H_{\mu\nu\lambda} - 2\mathcal{D}^\lambda\phi H_{\mu\nu\lambda}) + \mathcal{O}(\alpha'^2) \tag{4.2b}$$

If these equations are satisfied one finds that the central charge c of the worldsheet conformal field theory is indeed a constant and is given by

$$c = \frac{3}{2}\beta^\phi \tag{4.3}$$

where

$$\beta^\phi = D + \alpha'\Big(4(\mathcal{D}\phi)^2 - 4\mathcal{D}^2\phi - R + \frac{1}{2\cdot 3!}H^2\Big) + \mathcal{O}(\alpha'^2). \tag{4.4}$$

To construct string backgrounds, one needs a worldsheet CFT with $c = 15$. In the usual uncompactified superstring theory seen above, this is achieved by taking $D = 10$ and solving $\beta^g_{\mu\nu} = \beta^b_{\mu\nu} = 0$ and $\beta^\phi = 10$ at lowest order. For usual string compactification, $D = 4$ and $c = 6$. For weakly curved backgrounds, that is curvatures that are large compared to the string scale, this is a good approximation and the resulting equations of motion arise from the supergravity spacetime effective action. Furthermore we wish to take four dimensions to be large, nearly flat, and the remaining ones to be compact. For the α' expansion to be valid however the compact space must be smooth and have a length scale that is large compared with $\sqrt{\alpha'}$.

A way to avoid this is to consider string backgrounds which are a direct product of an internal, exact, CFT with $c_{\text{int}} = 9$ and a non-compact sigma model with a four-dimensional flat target space and hence $c_{\text{ext}} = 6$. A class of exact CFT's with $c_{\text{int}} = 9$ are provided by the so-called Gepner models and these are in turn constructed as a tensor product, with identifications, of minimal models. These will be discussed in detail in the following chapter.

Here we will try something slightly different. We will consider the CFT to be a tensor product of a sigma model with $D = 4$ with central charge $6 - \epsilon$ and an internal CFT with central charge $c_{\text{int}} = 9 + \epsilon$ where ϵ is a small dimensionless number, possibly negative. As a result we see that the conditions for conformal invariance of the non-compact sigma model are

$$4 - \frac{2\epsilon}{3} = 4 + \alpha'\Big(4(\mathcal{D}\phi)^2 - 4\mathcal{D}^2\phi - R + \frac{1}{2\cdot 3!}H^2\Big) + \mathcal{O}(\alpha'^2) \tag{4.5}$$

where the first equation comes from the condition $c = c_{\text{ext}} + c_{\text{int}} = 15$. From this equation we see that

$$\alpha' M^2 \sim \epsilon, \tag{4.6}$$

where again M is the scale of the curvature used to derive the beta functions for the fields in $D = 4$ background. Provided that $|\epsilon| \ll 1$ then the perturbation expansion is still valid, indeed we can think of the $\alpha' M^2$ expansion as an expansion in ϵ. We call ϵ the central charge deficit following [61] and our goal will be to search for models with small ϵ. We take the external theory to be in $D = 4$. Since the internal CFTs with $c_{\text{int}} = 9 + \epsilon$ will be constructed in the same spirit as Gepner models, we shall refer to them as ϵ-*Gepner models*.

Effective potential for the dilaton: The background equations of motion (4.2a),(4.2b),(4.5) can be derived as the Euler-Lagrange equations of the spacetime effective action

$$S_{\text{eff}} = \frac{1}{\alpha'} \int d^4X \sqrt{-g} e^{-2\phi} \left(R + 4(\mathcal{D}\phi)^2 - \frac{1}{2 \cdot 3!} H^2 - \frac{2\epsilon}{3\alpha'} \right) + \ldots, \tag{4.7}$$

where again the ellipsis denotes higher order terms in α' and derivatives, *i.e.* ϵ. To proceed it is useful to go to so-called Einstein frame

$$\tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu} \tag{4.8}$$

so that, in terms of $\tilde{g}_{\mu\nu}$ the spacetime action is

$$\tilde{S}_{\text{eff}} = \frac{1}{\alpha'} \int d^4X \sqrt{-\tilde{g}} \left(\tilde{R} - 2(\mathcal{D}\phi)^2 - \frac{1}{2 \cdot 3!} e^{-2\phi} H^2 - \frac{2\epsilon}{3\alpha'} e^{2\phi} \right) + \ldots \tag{4.9}$$

where all quantities with tilde are in Einstein frame. Here we find an effective dilaton potential

$$\tilde{V}(\phi) = \frac{2\epsilon}{3\alpha'} e^{2\phi} \tag{4.10}$$

We will be interested in cosmological solutions that are consistent with an FRW universe. In this case, it is helpful to dualise the NS-NS Kalb-Ramond field into an axion a via

$$H_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda\rho} e^{2\phi} \partial^\rho a \tag{4.11}$$

which accounts for another massless modulus. Thus we have

$$\tilde{S}_{\text{eff}} = \frac{1}{\alpha'} \int d^4X \sqrt{-\tilde{g}} \left(\tilde{R} - 2(\mathcal{D}\phi)^2 - \frac{1}{2} e^{2\phi} (\mathcal{D}a)^2 - \frac{2\epsilon}{3\alpha'} e^{2\phi} \right) + \ldots, \tag{4.12}$$

In addition to these universal string fields that arise from the NS-NS ground state of the string, a generic compactification will also lead to other light states, *i.e.* scalar field whose mass is small compared to $1/\sqrt{\alpha'}$. Thus the total effective action is

$$\tilde{S} = \tilde{S}_{\text{eff}} - \frac{1}{2\alpha'} \sum_a \int d^4X \sqrt{-\tilde{g}} (\partial_\mu \chi_a \partial^\mu \chi_a + e^{2\phi} m_a^2 \chi_a^2) \tag{4.13}$$

where the χ_a represent the other light scalar fields and m_a is the string-frame mass, as computed in the CFT. Later in the text, we relate this to the conformal dimension in the test case of an exact background containing linear dilaton in the external, noncompact sector. We also explicitly verify that the dilaton remains massless as well as the factor $-2/3$ in (4.12) in the the appendix section A.2.

Large volume interpretation: Although Gepner models are non-geometric, algebraic compactifications, correspondence to geometrical objects has been established for many of them; see [41] for a good review. The Gepner model $(3,3,3,3,3)$ can be related to a sigma model on a Calabi-Yau manifold, namely the quintic hypersurface in $\mathbb{C}P^4$. There is no geometrical interpretation of our ϵ-Gepner models, but we might hope that some ϵ-Gepner models can be thought of as exact CFTs for stringy-sized Calabi-Yau manifolds with fluxes. We can see some evidence of this by looking at the beta functions equations given above. Indeed if we assume there to be a geometrical correspondence between the internal space and some six-dimensional compact manifold $\mathcal{M}$, re-writing (4.4) and integrating over the compact directions, we obtain

$$\begin{aligned}\int_{\mathcal{M}}(\beta^\phi - D) &= \alpha' \int_{\mathcal{M}} e^{-4\phi}\{4(\mathcal{D}\phi)^2 - 4\mathcal{D}^2\phi - R + \frac{1}{12}H^2\} \\ &= \alpha' \int_{\mathcal{M}} e^{-4\phi}\{4(\mathcal{D}\phi)^2 - 2\mathcal{D}^2\phi - \frac{1}{6}H^2\} \\ &= \alpha' \int_{\mathcal{M}} e^{-2\phi}(\mathcal{D}^2 e^{-2\phi} - \frac{1}{6}H^2) \\ &= -\alpha' \int_{\mathcal{M}} \{(\mathcal{D}e^{-2\phi})^2 + \frac{1}{6}e^{-4\phi}H^2\} \leq 0 \end{aligned} \tag{4.14}$$

with equality holding for $H_{\mu\nu\lambda} = 0$ and $\mathcal{D}_\mu\phi = 0$. Note that to go from the first to the second line in this computation, we have used the g-beta function equation above and as discussed previously, β^ϕ is constant when equations for β^g and β^b are satisfied. That is, $\beta^\phi - D \leq 0$ or $c \leq 3D/2$. Since the equality $c = 3D/2$ also means that we have the usual Calabi-Yau compactification (in $D = 10$, $c = 15$), a non-zero $H_{\mu\nu\lambda}$ must mean a Calabi-Yau with fluxes. Note also that the argument is for the NS-NS fluxes since the Kalb-Ramond field $b_{\mu\nu}$, whose field strength is $H_{\mu\nu\lambda}$ is an NS-NS field. A similar result is expected to hold for the RR fields. Moreover, this argument is merely indicative as it does not take into account the warping of the metric in such compactifications.

4.2. Effective Spacetime Physics of Models with $\epsilon \neq 0$

As discussed above in the introductory sections, the geometric compactifications of the string theory target space are made up of an external manifold representing the four-dimensional, noncompact spacetime and a compact Calabi-Yau manifold whose choice influences the non-compact part. An alternative algebraic formulation of this compactification procedure may be carried out in terms of the underlying conformal field theories. The classic example of such CFT compactifications are Gepner Models [38]. These will be reviewed in their technical details in the next section. Briefly, these are typically thought of as CFT analogues of compact Calabi-Yau manifolds whose size is of order the string scale, however they are formulated without recourse to geometric data. The construction of Gepner Models involves taking a tensor product of minimal models whose central charges add up to 9.

The models that we discuss here carry the correct overall central charges $c = 15$, but the decomposition differs from the usual $6 + 9$ split for the internal and external sectors. Instead we take our external theory to be slightly non-flat with $c_{\text{ext}} = 6 - \epsilon$ and hence $c_{int} = 9 + \epsilon$, with $|\epsilon| \ll 1$. Other discussions of non-geometric compactifications and moduli stabilisation include [22], which will not be covered here.

We will see that there are plenty of explicit internal compact unitary CFT's with $|\epsilon| \ll 1$. For $\epsilon < 0$ there is no limit to how small ϵ can be. However for $\epsilon > 0$ this does not seem to be the case and the best models that we have found have $\epsilon \sim 10^{-6}$. Keeping ϵ small means that the theory remains weakly coupled and the external curvatures are small so one recovers a reliable four-dimensional effective field theory. Furthermore models can be obtained such that the spectrum of the full theory is free of massless moduli. The exact spectrum is model-dependent but we discuss general characteristics shared by all models constructed in the prescribed way. Although in this thesis, we will not seek to obtain a realistic spectrum of massless fields, we hope that this can be addressed in future work.

4.2.1. The $\epsilon > 0$ Models

For $\epsilon > 0$ we find that the dilaton will ultimately run to weak coupling; $\phi \to -\infty$ (Figure 4.1). However there will be solutions where ϕ initially runs up the potential, stops, and then rolls back down to $\phi \to -\infty$. At the turning point $V(\phi) > 0$ with $\dot{\phi} = 0$ and hence there will be a short period of inflation as in the model of [20] (only here it is realised within perturbative string theory).

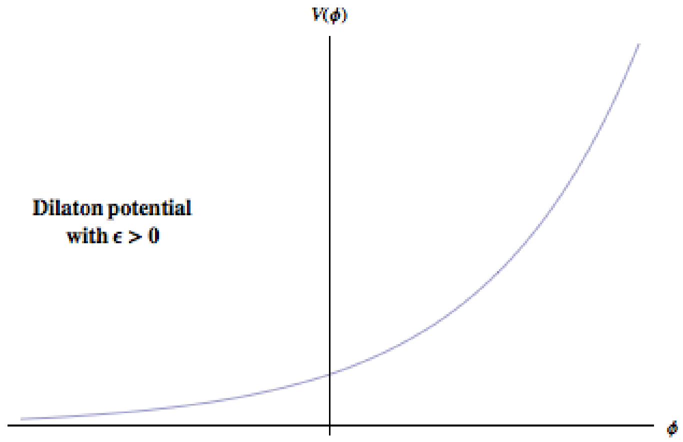


Figure 4.1: $\epsilon > 0$

For large ϕ the strings are strongly coupled and ϵ cannot be used to suppress the curvatures and our α' expansion isn't valid. For $\epsilon > 0$ this region can be avoided however.

4.2.2. The $\epsilon < 0$ Models

If $\epsilon < 0$ then we see that eventually $\phi \to \infty$ and we always end up in the uncontrolled regime (Figure 4.2). Suppose, however, that we introduce D-branes into the background which are

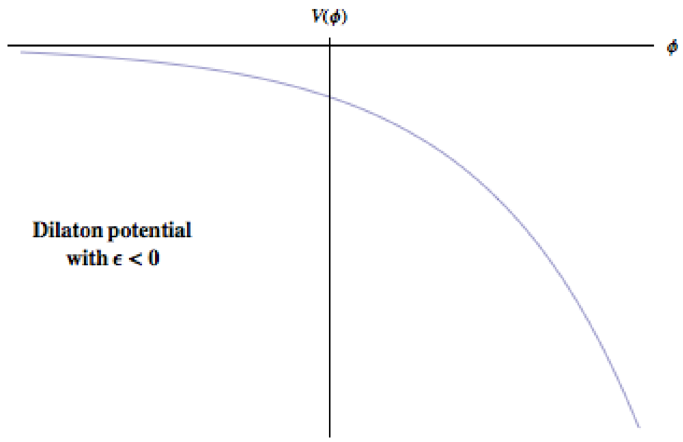


Figure 4.2: $\epsilon < 0$

extended along the four non-compact spatial directions. Such a brane would introduce a term in the effective potential of the form

$$S_{\text{brane}} = -T \int d^4x \sqrt{-g} e^{-\phi} \tag{4.15}$$

with $T > 0$ and again we are in the string frame. Transforming to the Einstein frame gives

$$\tilde{S}_{\text{brane}} = -T \int d^4x \sqrt{-\tilde{g}} e^{3\phi} \tag{4.16}$$

Combining this with the charge deficit contribution we arrive at the effective dilaton potential (Figure 4.3),

$$\tilde{V}(\phi) = \frac{2\epsilon}{3\alpha'} e^{2\phi} + T e^{3\phi} \tag{4.17}$$

This potential has an extremum at

$$\frac{4\epsilon}{3\alpha'} e^{2\phi} + 3T e^{3\phi} = 0 \tag{4.18}$$

which, for positive tension branes ($T > 0$), has a solution if $\epsilon < 0$;

$$e^{\phi_0} = -\frac{4\epsilon}{9T\alpha'} \tag{4.19}$$

with

$$\tilde{V}(\phi_0) = \frac{32\epsilon^3}{9^3 \alpha'^3 T^2} < 0 \tag{4.20}$$

One can see that this is in fact a stable minimum. For small and negative ϵ this critical point is weakly coupled with a small but negative cosmological constant.

In the rest of this chapter, we will develop the worldsheet CFT's required in these constructions. We first consider the case $\epsilon > 0$ where the running of the dilaton down the potential can be modelled exactly by a timelike linear dilaton CFT (although there are of course other solution whose exact CFT is unknown). For the $\epsilon < 0$ case we need to construct the internal compact CFT and also find suitable D-branes that can be introduced to stabilise the dilaton.

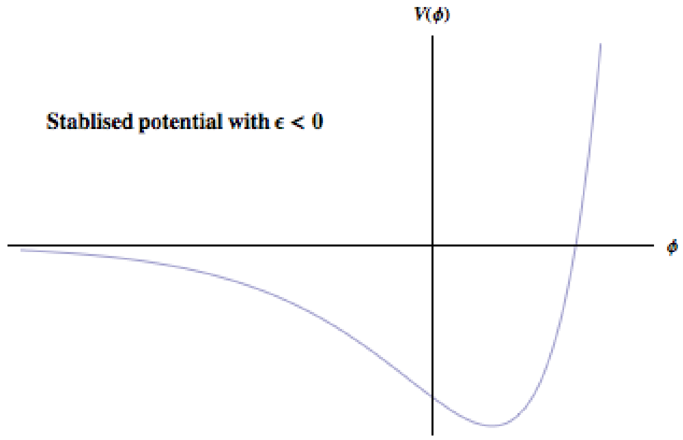


Figure 4.3: Stabilising the potential when $\epsilon < 0$

We will see that models can readily be found with small ϵ and no massless fields. In such cases there are no problems with tadpoles from the D-branes and thus they can be consistently incorporated into the background. Furthermore the absence of massless field implies that the only massless moduli left is the axion a. However typically non-perturbative such effects, such as worldsheet instantons, produce a periodic potential for a which must therefore have a global minimum. Thus these models have an AdS_4 vacuum with no massless fields.

In the following, we will present examples of CFTs that exhibit non-zero central charge deficits in the external and internal sector. One should keep in mind, however, that the non-trivial dilaton potential in principle affects the "trustworthiness" of these CFT backgrounds. For $\epsilon > 0$, the dilaton is driven towards $\phi_0 = -\infty$, thus string loops effects are switched off and the tree-level CFT may be regarded as a complete description of the background. For $\epsilon < 0$, on the other hand, adding a brane makes the dilaton settle at a small but finite value of g_S; thus the tree-level is not the full story, and loop corrections (as well as backreactions of the brane) will play a role. At the minimum of the dilaton potential, the theory should in principle be described by some new CFT taking these corrections into account, and due to the negative cosmological constant $V(\phi_0)$, it is natural to suspect this CFT to be realised by a sigma model with an AdS_4 target space. However a main benefit of our construction is that the derivative expansion turns out to be an expansion in ϵ and hence the leading order contributions to the effective theory can be very accurate. Thus in these cases we don't expect the exact solution to differ from that obtained from the effective action in any substantial way.

4.3. Suitability Conditions

As mentioned from the outset, these models are critical string theories with $c = 15$. We generally consider $\mathcal{N} = 1$ worldsheet superconformal field theories in the external sector to be able to implement GSO projection. $\mathcal{N} = 2$ supersymmetric theories may not be suitable for moduli stabilisation. The external sector must also have continuous degrees of freedom. The central feature of these models is that as long as the central charge deficit ϵ is kept small, the perturbation theory remains valid and therefore any consistent conformal field theories may be used as building blocks of the external and internal sectors. This being the only condition for our models affords a great amount of freedom for model building.

In this work, we have used known and well-understood conformal field theories, the minimal models in the internal sector. In principle however, this is not necessary and any rational theory may be inserted in the internal sector. Likewise in the external sector, we may use any CFT capable of providing the charge deficit with the opposite sign to the internal sector.

4.4. A Note on Similar Approaches

There have been numerous techniques developed to study noncritical string theories, that is theories with $c \neq 15$. Most of these begin with the same basic example as our models, namely the linear dilaton. Our models however are all critical string theories with nontrivial internal charge split between the external and the internal sectors.

At the start of this section, we briefly stated some previous approaches exploring the idea of a central charge deficit. Here, for completion's sake, we present a discussion of these techniques that share the basic premise of introducing the charge deficit when partitioning the internal and external sectors. We will examine the motivation for studying them, the outcomes as well as their shortcomings. Among the approaches approaches that have previously been explored, we shall focus on the de Alwis, Polchinski and Schimmrigk [59] and Antoniadis [61]. It should be also noted that, to the best of our knowledge, these lines of inquiry have not been pursued any further since the original works were published.

Discrete Inflation of Antoniadis, at al: The idea of a central charge deficit was used to study time-dependent backgrounds by the authors of [61]. Indeed the search for time-dependent cosmological solutions is the main motivation for them. They wish to realise solutions of the Robertson-Walker form in the external sector of the theory and find that these solutions contribute a central charge of less than four, $c_{RW} = 4 - \delta c$, with the central charge deficit $\delta c > 0$ in their models. Since the overall central charge must still be 15, they consider internal theories with the charge deficit with opposite sign. These are taken to be minimal models whose charges add up to slightly above 9. The discrete nature of these internal theory charges induces what they call "discrete inflation" in their models.

Starting with the beta functions and demanding that the total central charge must be 26 in the bosonic case, they find

$$\delta c = -3e^{-\phi}\{R + \Box\phi + \frac{1}{2}(D\phi)^2 - \frac{1}{2}e^{2\phi}(Db)^2\} \tag{4.21}$$

in units of $\alpha' = 1$. They also find that the effective action now contains a potential that depends on the charge deficit

$$V = \frac{1}{3}\sqrt{-g}\, e^{\phi}\delta c. \tag{4.22}$$

Note that this is analogous to writing ϵ in terms of V^2 in our models. The underlying metric of the Robertson-Walker universe is given by $ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j$, where $\tilde{g}_{ij}$ is a maximally symmetric metric of a 3-dimensional space and $a(t)$ is the scale factor. Using this metric, they can write the central charge deficit in terms of the Hubble parameter $H(t) = \dot{a}(t)/a(t)$

$$\delta c = 6e^{-\phi}(\dot{H} + 3H^2 + 2K/a^2), \tag{4.23}$$

where K is the constant curvature of the maximally symmetric three-space. Since, for the non-flat case of interest, $\delta c \neq 0$, the above may be solved for the dilaton

$$\phi = \ln\left(\frac{6}{\delta c}(\dot{H} + 3H^2 + 2K/a^2)\right). \tag{4.24}$$

By using the beta function equation for ϕ and solving for H, they find that in the absence of a potential for the dilaton, the noncompact part may only have two possible solutions, both with a non-negative curvature K. The trivial solution is $H = 0$, which leads to $\delta c = 12e^{-\phi_0}K$. This is the *static Einstein universe.* The second solution is $H = 1/t$ with $\delta c = 12e^{-\lambda}(1+K)$, where λ is an integration constant. They term this solution the *Milne universe.* Moreover they find these to two to be the only asymptotic solutions of the σ-model equations in the Robertson-Walker background metric. They further show that these remain solutions to all orders by identifying the corresponding conformal field theories. For the static Einstein universe, it is a Wess-Zumino-Witten model on an O(3) group manifold, with a free worldsheet parameter as the time coordinate. For the Milne universe, it turns out to be a GKO coset model with coset group $O(1,4)/O(1,3)$. A third solution in these models appears for a non-zero dilaton potential, which may be generated by string loop effects or by non-perturbative effects. In this case, they find the Hubble constant during inflation is determined by the dilaton field value. Out of the possible solutions, in the cases where there are known conformal field theories, they find that there are only finite number of Robertson-Walker cosmologies with central charges less than 4 in the external space. Note that they work our the spectrum of low-lying string excitations using a mass formula which differs from the one we derive in section 5.1.2 from the effective action. Our derivation coincides with Chamseddine [60] and is further supported by computing quantum fluctuations in the linear dilaton background.

Models of de Alwis, Polchinski and Schimmrigk: In a separate use of the central charge deficit idea, de Alwis, Polchinski and Schimmrigk [59] have a two-fold objective for considering such models. Firstly, they wish to study the vanishing of the cosmological constant by constructing non-trivial heterotic and superstring backgrounds. In addition, they are interested in using the idea of a nonzero ϵ to produce small tree level breaking of supersymmetry. Models they consider consist of a four-torus, an $\mathcal{N} = 2$ minimal model and an E(8) $\times$ SO(10) current algebra. They abandon this construction as a viable model because taking $k \to \infty$ for the level of any of the minimal models to obtain a small charge deficit leads to decompactification of the internal sector. We find this not be an issue as there is no need to take such a limit. Indeed in the candidate models we consider here, the tensor product of minimal models with smallest $|\epsilon|$ in the $\epsilon > 0$ sector has the highest k of 1805. In the $\epsilon < 0$ sector, taking a single relatively large k is found to be sufficient.

These models were aimed at solving problems other than moduli stabilisation, the issue that is of main concern to us. They also predate some key developments in string theory such as D-branes that help in particular to stabilise the dilaton for $\epsilon \leq 0$. We also carry out a detailed analysis of the exact spectrum via Gepner model techniques.

Chapter 5

The Noncompact Conformal Fields Theories

The target space manifold $\mathfrak{M}$ and the corresponding $c = 15$ conformal field theory $\mathfrak{C}$ are both split into internal and external sectors. From the target space perspective, the external or noncompact manifold $\mathfrak{M}_{\text{ext}}$ of our models represents the four dimensional world we live in. The corresponding worldsheet theory in the external sector, $\mathfrak{C}_{\text{ext}}$ is made up of a nonrational conformal field theory with continuous spectrum.

As outlined in section 4.3, in principle the models discussed here would admit any valid nonrational CFT. For physical reasons, we will be interested in a supersymmetric theory. The additional condition we will demand on any candidate $\mathfrak{C}_{\text{ext}}$ is that its central charge $c_{\text{ext}} = 6 - \epsilon$ must allow for small charge deficit ϵ as stipulated above for the perturbative beta function equations to remain valid. We will also demand some supersymmetry in order to have a consistent tachyon-free string background. These conditions will be further elaborated upon below.

As an example of such a CFT, we will consider the linear dilaton. For most purposes, this will be nothing more than a toy model used because most aspects of it are well understood. We will also look at another exactly soluble, but non-trivial example, namely Liouville theory. We will begin with the bosonic and $\mathcal{N} = 1$ versions of this theory, although these will not be of direct physical relevance to us in model building as they do not provide small enough charge deficit. We will also present the work done on the analytic continuation of the $\mathcal{N}=1$ theory since it was initially believed to be a good candidate for the time dependent backgrounds. The $\mathcal{N}=2$ case will be more suitable for our purposes and will be discussed briefly at the end of this chapter.

5.1. The Linear Dilaton CFT

The linear dilaton is one of the simplest examples of a conformal field theory. Although the linear realisation of the dilaton $\phi(X) = V_\mu X^\mu$ manifestly breaks Lorentz invariance, it will nonetheless serve as a useful model for studying the salient features of the string backgrounds we wish to construct. In this section, we attempt to survey the linear dilaton background, outline its limitations, collect its worldsheet data and understand it from the supergravity perspective. The key piece of information we will need later for our models is the mass shell formula arising from the physical state condition in string theory.

5.1.1. The Basic Data

In section 2.1.3, we have seen that a coupling to a generic dilaton $\phi(X)$ breaks the conformal invariance. In critical dimension $D = 26$, the first term in $\beta(\phi)$ vanishes. For $D \neq 26$, a nontrivial solution

$$\phi(X) = V_\mu X^\mu, \qquad V \cdot V = V_\mu V^\mu = \frac{26 - D}{6\alpha'} \tag{5.1}$$

is (2.33) discussed in [56]. Here V_μ is a constant background vector and X^μ are the bosonic coordinates. The corresponding worldsheet CFT is called the linear dilaton CFT. The energy momentum tensor

$$T_{\alpha\beta} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}} \tag{5.2}$$

for a bosonic linear dilaton CFT is given by

$$T_{\alpha\beta} = -\frac{1}{\alpha'} \partial_\alpha X^\mu \partial_\beta X_\mu + V_\mu \partial_\alpha \partial_\beta X^\mu \tag{5.3}$$

where we have dropped the explicit normal ordering symbols. The indices α, β run over the worldsheet coordinates τ, σ. We can rewrite these in terms of complex variables $z, \bar{z}$ [11]

$$T(z) = -\frac{1}{\alpha'} :\partial X_\mu \partial X^\mu: + V_\mu \partial^2 X^\mu, \tag{5.4a}$$

$$\bar{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X_\mu \bar{\partial} X^\mu: + V_\mu \bar{\partial}^2 X^\mu. \tag{5.4b}$$

The TT OPE may be computed as follows

$$\begin{aligned} T(z)T(w) = & \frac{1}{\alpha'^2} :\partial X_\mu(z)\partial X^\mu(z)::\partial X_\nu(w)\partial X^\nu(w): \\ & - \frac{1}{\alpha'} :\partial X_\mu(z)\partial X^\mu(z): V_\nu \partial^2 X^\nu(w) \\ & - \frac{1}{\alpha'} :\partial X_\nu(z)\partial X^\nu: V_\mu \partial^2 X^\mu + V_\mu V_\nu \partial^2 X^\mu \partial^2 X^\nu \\ & \sim (D + 6\alpha' V \cdot V)\frac{1}{2(z-w)^4} + \mathcal{O}\Big(\frac{1}{(z-w)^2}\Big). \end{aligned}$$

The central charge of this CFT may be read off from this as

$$c = D + 6\alpha' V \cdot V. \tag{5.5}$$

Comparing this expression to the dilaton beta function (2.32) and noting that the critical dimension of bosonic string theory is $D = 26$, we can classify the theories based on the norm of the background vector as follows,

V lightlike,	$V \cdot V = 0$	$\Rightarrow$	$c = 26$:	critical string theory,
V spacelike,	$V \cdot V > 0$	$\Rightarrow$	$c > 26$:	supercritical string theory,
V timelike,	$V \cdot V < 0$	$\Rightarrow$	$c < 26$:	subcritical string theory.

A good review of non-critical string theories was presented in [60]. Note that in our models, we always take the dilaton to lie in 4 external spacetime dimensions. The central charge in the bosonic version of such models will therefore be $4 + 6\alpha' V \cdot V$ and the above three cases critical, subcritical and supercritical theories will correspond to central charges $c = 4$, $c > 4$ and $c < 4$ respectively. The full CFT including both the internal and external sector in our constructions will always provide a critical string background with $c = 26$ (or $c = 15$ in the supersymmetric case) and therefore, we shall not use the terms critical, subcritical or supercritical when referring to such models. Instead, these will be referred to as $\epsilon = 0$, $\epsilon < 0$ and $\epsilon > 0$ branches respectively.

Indeed we shall mainly use the supersymmetric version of this argument where, the bosons X^μ are accompanied by free fermions ψ^μ, and the generating fields of the (left-moving) $N = 1$ superconformal algebra can be written as

$$
\begin{aligned}
T_{\text{ext}}(z) &= -\frac{1}{\alpha'}\,{:}\partial X_\mu \partial X^\mu{:} + V_\mu \partial^2 X^\mu - \frac{1}{2}\,\psi_\mu \partial \psi^\mu, \\
G_{\text{ext}}(z) &= i(2/\alpha')^{\frac{1}{2}}\,\psi_\mu \partial X^\mu - i(2\alpha')^{\frac{1}{2}}\,V_\mu \partial \psi^\mu
\end{aligned}
\tag{5.6}
$$

The central charge of the super-Virasoro algebra is, in $D = 4$ dimensions,

$$
c_{\text{ext}} = 6 + 6\alpha' V^2 \ , \tag{5.7}
$$

so unless V_μ is lightlike, the linear dilaton can serve as the external CFT in string compactifications with non-trivial central charge deficit. V_μ is constrained by the choice we will make for ϵ in the internal CFT,

$$
\epsilon = -6\alpha' V_\mu V^\mu \ . \tag{5.8}
$$

In many respects, the linear dilaton is close to a free boson CFT, the main difference being the V_μ-term in $T_{\text{ext}}(z)$. In a free scalar field theory, the operator ∂X^μ is a conformal primary with conformal dimension $(1, 0)$, but in a linear dilaton CFT it has different properties under conformal transformation since the energy-momentum tensor has now been modified. Let us compute this explicitly,

$$
\begin{aligned}
T(z)\partial X^\mu(w) &= -\frac{1}{\alpha'}\,{:}\partial X_\nu(z)\partial X^\nu(z){:}\,\partial X^\mu(w) + V_\nu \partial^2 X^\nu(z)\partial X^\mu(w) \\
&= \frac{-2}{\alpha'}\left(\frac{-\alpha'}{2(z-w)^2}\partial X^\nu \eta_{\mu\nu} + \text{reg} + V_\nu \eta^{\mu\nu}\frac{\alpha'}{(z-w)^3}\right).
\end{aligned}
$$

Due to the $(z - w)^{-3}$ term, ∂X^μ is not primary, but only quasi-primary. As a consequence, the field ∂X^μ is not a conserved current, instead there is a so-called background charge, and correlation functions would have to be computed using screening operators, see [37] and also

[61].

The fields we are interested in for string theory purposes are the vertex operators

$$\mathcal{V}_p =: e^{ip_\mu X^\mu} :, \tag{5.9}$$

where p_μ is a 4-vector. We will need their OPEs with the energy-momentum in order to determine their conformal dimensions

$$\begin{aligned}
T(z)\mathcal{V}_p(w) &= -\frac{1}{\alpha'} :\partial X_\mu(z)\partial X^\mu(z) :: e^{ip_\nu X^\nu(w)} : + V_\mu \partial^2 X^\mu(z) : e^{ip_\nu X^\nu(w)} : \\
&= -\frac{\alpha'}{2}\sum_{n=0}^{\infty}\frac{(ip_\nu)^n}{n!} :\partial X_\mu(z)\partial X^\mu(z) :: X^\nu(w)^n : \\
&+ V_\mu \sum_{n=0}^{\infty}\frac{(ip_\nu)^n}{n!}\partial^2 X^\mu(z) : X^\nu(w)^n : \\
&= \alpha' \frac{p^2}{4(z-w)^2}\mathcal{V}_p(w) + i\alpha' \frac{V_\mu p^\mu}{2(z-w)^2}\mathcal{V}_p(w) + \text{reg.}
\end{aligned}$$

Hence $\mathcal{V}_p(z)$ is a primary field with complex conformal dimension

$$h_p = \frac{\alpha'}{4}(\, p_\mu p^\mu + 2ip_\mu V^\mu) \ . \tag{5.10}$$

The translational invariance is broken by a linear dilaton and the dilaton couples to the Euler characteristic χ, the n-point correlator $\langle \mathcal{V}_{p_1}\mathcal{V}_{p_2}\ldots\mathcal{V}_{p_n}\rangle$ is only non-vanishing if the momenta satisfy (see Myers [56])

$$\sum_{j=1}^{n} p_j = -iV\chi. \tag{5.11}$$

This is known as the *charge neutrality condition* and on the sphere where $\chi = 2$, it reduces to $\sum_{j=1}^{n} p_j = -2iV$. Note that here we have dropped the spacetime indices μ from the momenta. For a vertex operator $\mathcal{V}_p$, the 'dual' or 'conjugate' field is a field $\mathcal{V}_p'$ with the same conformal dimension as $\mathcal{V}_p$ and $\langle \mathcal{V}_p\mathcal{V}_{p'}\rangle \neq 0$. The two-point function of $\mathcal{V}_p$ with itself would violate the charge neutrality condition (5.11). On the other hand, we note that the operator $\mathcal{V}_{-p-2iV}$ has the same dimension as $\mathcal{V}_p$

$$h_{-p-2iV} = \frac{\alpha'}{4}\{(-p-2iV)^2 + 2iV(-p-2iV)\} = \frac{\alpha'}{4}(p^2 + 2iVp).$$

The two-point function of $\mathcal{V}_p$ with $\mathcal{V}_{-p-2iV}$ does not violate (5.11) and $\mathcal{V}_{-p-2iV}$ is therefore the conjugate field of $\mathcal{V}_p$.

5.1.2. The Mass Formula

The physical state condition: For the bosonic linear dilaton CFT, the Virasoro generators are given by [11]

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha^\mu_{m-n}\alpha_{\mu n}: + i\sqrt{\frac{\alpha'}{2}}(m+1)V_\mu p^\mu_m. \tag{5.12}$$

The eigenvalues of N are $0, 1, 2, \ldots$. In order to analyse spectra of string compactifications, one starts from the physical state condition given in terms of the zero mode of the total energy-momentum tensor

$$(\, L_0^{\mathrm{tot}} - a \,)\, |\mathrm{phys}\rangle = 0 \tag{5.13}$$

to extract a formula for the mass of a string state in terms of conformal dimensions. a is a normal-ordering constant.We tacitly assume that $|\,\mathrm{phys}\rangle$ is a primary state for the super-Virasoro algebra to begin with.

The total zero mode L_0^{tot} is a sum of the internal L_0^{int} and of the zero mode of the linear dilaton energy-momentum tensor which acts as

$$L_0^{\mathrm{ext}} = \frac{\alpha'}{4}p_\mu p^\mu + i\frac{\alpha'}{2}V_\mu p^\mu + N\ . \tag{5.14}$$

Here, the offset N from the conformal dimension (5.10) accounts for modes L_n^{ext} or ψ_r^μ applied to ground states $|p\rangle$; we have $N \in \frac{1}{2}\mathbb{Z}_+$ in the Neveu-Schwarz sector and $N \in \mathbb{Z}_+$ in the Ramond sector.

Inserting this into the physical state condition, we get

$$\frac{\alpha'}{4}p_\mu p^\mu + i\frac{\alpha'}{2}V_\mu p^\mu = \Delta_{N,a} \qquad \text{with} \qquad \Delta_{N,a} = a - N - h_{\rm int} \tag{5.15}$$

and see that we need to admit complex

$$p^\mu = \rho^\mu + i\sigma^\mu \tag{5.16}$$

in order to have non-trivial solutions – assuming that the conformal dimensions $h_{\rm int}$ from the internal CFT are real. This is also clear from the charge neutrality condition (5.11). Substituting this in the physical state condition, we get

$$\begin{aligned}&\frac{\alpha'}{4}\Big\{-(\rho^0+i\sigma^0)^2+(\vec{\rho}+i\vec{\sigma})^2\Big\}\\ &\qquad\qquad -i\frac{\alpha'}{2}\Big\{V^0(\rho^0+i\sigma^0)+i\vec{V}(\vec{\rho}+i\vec{\sigma})\Big\}+N+h_{\rm int}=a\end{aligned}$$

or

$$\frac{\alpha'}{4}\Big\{\rho_\mu\rho^\mu - \sigma_\mu\sigma^\mu + 2i\rho_\mu\sigma^\mu\Big\} + \frac{\alpha'}{2}\Big\{iV_\mu\rho^\mu - V_\mu\sigma^\mu\Big\} + N + h_{\rm int} = a. \tag{5.17}$$

By separating out the real and imaginary parts of this equation

$$\frac{\alpha'}{4}\rho_\mu\rho^\mu - \frac{\alpha'}{4}\sigma_\mu\sigma^\mu - \frac{\alpha'}{2}V_\mu\sigma^\mu + N + h_{\rm int} = a \tag{5.18a}$$

$$i\frac{\alpha'}{2}\rho_\mu\sigma^\mu + i\frac{\alpha'}{2}\rho_\mu V^\mu = 0, \tag{5.18b}$$

we see that the imaginary part of the equation is solved by fixing $\sigma_\mu = -V_\mu$, although this is not a unique solution. Inserting this into the real part and using (5.8), we can rewrite the physical state condition in terms of the charge deficit ϵ as

$$\frac{\alpha'}{4}\rho_\mu\rho^\mu = \Delta_{N,a} + \frac{\epsilon}{24}\ . \tag{5.19}$$

At this point, we face certain choices for the notion of mass as well as the normal ordering constant a. Our choice of the mass formula is in line with Chamseddine [60]. Other authors interpret this in a different way – a critique of their approaches was presented in section 4.4. We relate the spacetime mass of a string state to the quantities in the physical state condition. We will set

$$m^2 \equiv -\frac{\alpha'}{4}\left(p_\mu p^\mu + 2iV_\mu p^\mu\right) = -\Delta_{N,a} \tag{5.20}$$

This choice is motivated by considering the coupling of the linear dilaton to a massive scalar field [60] with the action given by

$$-\frac{1}{2\alpha'}\int d^4x\sqrt{-g}\, e^{-2\phi}\{(\partial\chi)^2 + m^2\chi^2\}, \tag{5.21}$$

whose equation of motion gives the mass-shell condition

$$p_\mu p^\mu + 2iV_\mu p^\mu + m^2 = 0. \tag{5.22}$$

Thus we can identify m^2 in the effective action with the mass-squared computed by the conformal dimension of the internal CFT via (5.20). By computing quantum fluctuations around the linear dilaton background in the weak gravity limit, we can see that the dilaton and the graviton remain massless, even when the background affects the physical mass via the background vector V_μ in the physical state condition. These computations are presented in the appendix A.2.

In the paper [61] – a summary of which was presented in section 4.4 – the other natural choice $m^2 = -\frac{\alpha'}{4}\rho_\mu\rho^\mu$ was used, which leads to an ϵ-dependent mass shift of the whole spectrum due to the presence of the linear dilaton. We will briefly come back to this choice in the next section, but let us point out right away that the most important qualitative feature of our string compactifications does not depend on which of the two definitions of spacetime mass we use: either way, we will encounter many models that have no massless moduli. Physically the difference arises because the background geometry is not flat and hence the solution to a massive wave equation can behave with a different mass from that which appears in the Lagrangian.

The second interpretational issue concerns the normal ordering constant a in the physical state condition $(L_0 - a)|\text{phys}\rangle = 0$, the choice of which directly influences the physical spectrum of the theory. Here, we will make the standard choices

$$a_{\rm NS} = \frac{1}{2}, \qquad a_{\rm R} = \frac{1}{2} - \frac{d}{16} \tag{5.23}$$

for the NS and R sectors respectively, where $d/16$ is the contribution coming from $d = D - 2$ spin fields in the transverse direction. This leads to $a_{\rm R} = \frac{3}{8}$ for our $D = 4$ linear dilaton theory.

The work [56], on the other hand, introduces an ϵ-shift into the normal ordering constant. There the shift arises from solving the light-cone gauge constraint, after having introduced non-standard V_μ-dependent Lorentz group generators, which allow to "restore" the Lorentz symmetry which is, at the face of it, broken by the linear dilaton theory. The effect of the shift in a on the physical state condition is as switching between the above definitions of mass.

5.1.3. The Boundary Theory – D-branes in Linear Dilaton

For the analysis of the open string sector, we wish to be able to impose Neumann boundary conditions in order to obtain spacetime filling branes. For the linear dilaton, these were studied by Bershadsky and Kutasov [49]. Note that Dirichlet boundary conditions were considered by Li [50].

5.2. Liouville Theory

Liouville theory is a rich subject with wide-ranging applications in string theory and low-dimensional quantum gravity. Here we are interested in the quantisation of Liouville theory as a CFT, implying that the space of states forms a representation of the Virasoro algebra. Since, the Liouville zero more $\int_0^{2\pi} d\sigma \Phi(\sigma)$ takes values in a noncompact space, we obtain a continuous set of representations [64]. This is a review largely of the spacelike bosonic Liouville theory. The $\mathcal{N} = 1$ formulae used for the timelike extension are based on [73] and [76] with certain choices of shift parameters motivated by requirements of analytic continuation in the next chapter. These are also in line with [80]. The $\mathcal{N} = 2$ conventions are based on [77] where we also give some original computations to show modular invariance of the $\mathcal{N} = 2$ partition function in the NS sector of the bulk theory.

5.2.1. The Bosonic Case

The ordinary spacelike Liouville theory with a central charge $c = 1 + 6Q^2$ is given by the following action defined on a closed surface [72]

$$S_L(b, \mu) = \frac{1}{4\pi} \int d^2\sigma \sqrt{g}\big((\nabla\phi)^2 + QR\phi + 4\pi\mu e^{2b\phi}\big). \tag{5.24}$$

Here $\mu > 0$ and $b \in \mathbb{R} \setminus \{0\}$ is the central parameter of the theory. For the interaction $\mu e^{2b\phi}$ to be a marginal perturbation of the linear dilaton CFT, it is required that

$$Q = b + \frac{1}{b} \tag{5.25}$$

This implies that $|Q| \geq 2$ and thus $c \geq 25$ for the spacelike Liouville. The Liouville mode may be thought of as a spatial target space coordinate leading to a non-trivial string background [44]. The vertex operators in this theory are given by

$$V_a = e^{2a\phi}, \quad a \in \mathbb{C}. \tag{5.26}$$

A state $|a\rangle$ may be constructed by acting on the vacuum by a vertex operator V_a such that $|a\rangle = V_a|0\rangle$. The conformal dimension of such a vertex operator is

$$h_a = a(Q - a). \tag{5.27}$$

This operator is the 'dual' operator to V_{Q-a} and it has $h_{Q-a} = a(Q - a)$.

Note that the zero-mode quantum mechanics picture of Liouville theory, also known as the minisuperspace analysis in the literature, is a particularly illuminating way of looking at the spectrum of both the spacelike and the timelike Liouville theories. We refer the reader to sections 2 and 3 of [72].

This theory has a duality symmetry $b \to b^{-1}$ which follows from the condition of crossing symmetry of the four-point functions that requires the theory be invariant under a dual interaction $\tilde{\mu}e^{2\phi/b}$ with μ and $\tilde{\mu}$ related by $\{\pi\mu\gamma(b^2)\}^{b^{-1}} = \{\pi\tilde{\mu}\gamma(b^{-2})\}^{b}$. The function $\gamma(x)$ will be frequently used in our computations on Liouville theory and is given by a quotient of ordinary Gamma functions

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{5.28}$$

On the sphere, the three-point functions

$$\langle V_{a_3}(z_3)V_{a_2}(z_2)V_{a_1}(z_1)\rangle = \frac{C_{\text{sl.}}(a_3,a_2,a_1|b)}{(z_{12}\bar{z}_{12})^{h_{a_1}+h_{a_2}-h_{a_3}}(z_{23}\bar{z}_{23})^{h_{a_2}+h_{a_3}-h_{a_1}}(z_{31}\bar{z}_{31})^{h_{a_3}+h_{a_1}-h_{a_2}}} \tag{5.29}$$

of this CFT are characterised by coefficients $C_{\text{sl.}}(a_3,a_2,a_1|b)$.

Shift relations: The above coefficients $C_{\text{sl.}}(a_3,a_2,a_1|b)$ may be determined uniquely via conformal bootstrap, a procedure that imposes the crossing symmetry on the four-point functions and leads to the following unique shift relations

$$\frac{C_{\text{sl.}}(a_3,a_2,a_1+b|b)}{C_{\text{sl.}}(a_3,a_2,a_1|b)} = H_{\text{sl.}}(a_3,a_2,a_1|b), \tag{5.30}$$

with

$$H_{\text{sl.}}(a_3,a_2,a_1|b) = \frac{1}{b^4}\gamma(2a_1b)\,\gamma(2a_1b+b^2)\frac{\gamma(\hat{a}b-2a_1b-b^2)\,\gamma(2-\hat{a}b+b^2)}{\gamma(\hat{a}b-2a_2b)\,\gamma(\hat{a}b-2a_3b)} \tag{5.31}$$

There is also a dual relation for $b \to b^{-1}$. The seminal works of DOZZ [65, 66] and Teschner [67] have led to the following expression for the coefficients $C_{\text{sl.}}$

$$C_{\text{sl.}}(a_3,a_2,a_1|b) = \left(\pi\mu\gamma(b^2)b^{2(1-b^2)}\right)^{(Q-\hat{a})/b}\frac{\Upsilon_b(b)}{\Upsilon_b(\hat{a}-Q)}\prod_j\frac{\Upsilon_b(2a_j)}{\Upsilon_b(\hat{a}_j)} \tag{5.32}$$

where $\hat{a} = \sum_j a_j$, $\hat{a}_j = \hat{a} - 2a_j$ and $\Upsilon_b(z)$ is an entire function defined in terms of the Barnes' double Gamma function Γ_2

$$\Upsilon_b(z) = \Gamma_2(z|b,b^{-1})^{-1}\,\Gamma_2(Q-z|b,b^{-1})^{-1} \tag{5.33}$$

Equation (5.32) is the so-called 'DOZZ formula'. Since we are extending McElgin's recipe [72], we will largely use his convention of writing $C_{\mathrm{sl.}}$ in terms of generic constants $A_{\mathrm{sl.}}$, usualy left undetermined by the bootstrap

$$C_{\mathrm{sl.}}(a_3, a_2, a_1|b) = A_{\mathrm{sl.}} \times \left(b^{2(b^{-1}-b)}\right)^{(Q-\hat{a})} \times \frac{\Upsilon_b(b)}{\Upsilon_b(\hat{a}-Q)} \prod_j \frac{\Upsilon_b(2a_j)}{\Upsilon_b(\hat{a}_j)} \tag{5.34}$$

The DOZZ formula (5.32) may be obtained by setting $A_{\mathrm{sl.}} = \pi\mu\gamma(b^2)$ and rescaling the vertex operators $V_a \to A_{\mathrm{sl.}}^{-a/Q} V_a$.

Using property of the three-point function that $C_{\mathrm{sl.}}(a_3, a_2, a_1|b) = C_{\mathrm{sl.}}(a_3, a_2, a_1|b^{-1})$ due to the duality $b \to b^{-1}$, it can be seen [72] that (5.34) is a unique solution to the shift relations (5.30). Uniqueness of the solution to shift relations follows from the fact that the ratio of any other solution to the one in (5.34) must be periodic in b and b^{-1} and therefore must be independent of all a_j since $b \in \mathbb{R}$ in the ordinary Liouville theory.

For the properties of the double Gamma function, please see appendices in [70]. Here we note that

$$\Upsilon_b(z) = \Upsilon_b(Q-z) = \Upsilon_{b^{-1}}(z) \tag{5.35}$$

and it has an integral representation

$$\ln \Upsilon_b(z) = \int_0^\infty \frac{dt}{t}\left\{\left(\frac{Q}{2}-z\right)^2 e^{-2t} - \frac{\sinh\left(\frac{Q}{2}-z\right)t}{\sinh(bt)\sinh(t/b)}\right\}. \tag{5.36}$$

A key property of $\Upsilon_b(z)$ needed later in this section is that it can be analytically continued to the entire complex b^2 plane except the negative real axis. Using the identities $\Upsilon_b(z+b) = \gamma(bz)b^{1-2bz}\Upsilon_b(z)$ and $\Upsilon_b(z+b^{-1}) = \gamma(zb^{-1})b^{2zb^{-1}-1}\Upsilon_b(z)$ the following reflection symmetry may be derived from the three-point function

$$V_a = R_{\mathrm{sl.}}(a) V_{Q-a} \tag{5.37}$$

where the reflection coefficient $R_{\text{sl.}}(a)$ satisfies $R_{\text{sl.}}(a)R_{\text{sl.}}(Q-a)=1$ and is given by

$$R_{\text{sl.}}(a)=(\pi\mu\gamma(b^2))^{(Q-2a)/b}\frac{\gamma(2ab-b^2)}{b^2\gamma(2-2ab^{-1}+b^{-2})}. \tag{5.38}$$

The term 'reflection' is due to the interpretation as reflection into the potential wall induced by the Liouville interaction [64]. The coefficients $C_{\text{sl.}}(a_3,a_2,a_1)$ too satisfy the following reflection relations

$$C_{\text{sl.}}(a_3,a_2,a_1)=R_{\text{sl.}}(a_3)C_{\text{sl.}}(Q-a_3,a_2,a_1). \tag{5.39}$$

Normalisation of States and Absence of the Vacuum $|0\rangle$*:* For a physical unitary theory, we must find a sensible way of defining a normalisation of states. We start by defining the scalar product of two states $|a_1\rangle$ and $|a_2\rangle$ to be the two-point function, which is in turn obtained from a three-point function in the following manner [64]

$$\begin{aligned}{}_{\text{in}}\langle a_2|a_1\rangle_{\text{in}} &= \lim_{a\to 0} C_{\text{sl.}}(a_2^*,a,a_1) \\ &\simeq \frac{2aR_{\text{sl.}}(a_1)}{(a_2-a_1+a)(a_1-a_2+a)} \\ &\qquad + \frac{2a}{(Q-a_2+a_1+a)(a_1+a_2-Q+a)},\end{aligned} \tag{5.40}$$

where $a^*=Q-a$. Symmetry and analyticity properties of the DOZZ formula (5.32) leads to the bound

$$0<\text{Re}(a_j)\leq\frac{Q}{2},\quad i=1,2,3. \tag{5.41}$$

See [64] for details. The coefficient $C_{\text{sl.}}(a_2^*,a,a_1)$ in (5.40) is non-vanishing in the limit $a\to 0$ only when $a_2^*=a_1$ or $a_2^*=Q-a_1$. Note that, normalisation cannot be defined in the strict sense of $\langle a|a\rangle=1$, but only in a distributional sense $\langle a|a\rangle\sim\delta(\dots)$.

From equation (5.40), we can also draw conclusion about the unitarity of the theory. One obtains complex norm square unless

$$a\in\mathbb{R}\qquad\text{or}\qquad a\in\frac{Q}{2}+ip,\ p\in\mathbb{R} \tag{5.42}$$

For the $a_i \in \mathbb{R}$ case, one would have to choose in (5.40) an a with $\operatorname{Im} a \neq 0$, which would lead to

$$ {}_{\text{in}}\langle a_2|a_1\rangle_{\text{in}} = \pm 2\pi i R_{\text{sl.}}(a_1)\delta(a_2 - a_1). \tag{5.43}$$

Since we want a real scalar product, this does not give a satisfactory answer when $a_i \in \mathbb{R}$. The other branch $a = \frac{Q}{2} + ip$ is more promising. The reflection relation (5.39) allows us to restrict to $p > 0$. Taking $\operatorname{Re}(a_j) > 0$ as required by (5.41) and letting $|p_j\rangle \equiv |\frac{Q}{2} + ip_j\rangle$ means

$$ {}_{\text{in}}\langle p_2|p_1\rangle_{\text{in}} = 2\pi\delta(p_2 - p_1). \tag{5.44}$$

We can state (5.27) as follows

$$h_a = h_{\frac{Q}{2}+ip} = \left(\frac{Q}{2} + ip\right)\left(Q - \frac{Q}{2} - ip\right) = \frac{Q^2}{4} + p^2 > \frac{Q^2}{4}. \tag{5.45}$$

Note that neither the $|p = 0\rangle$ state nor the vacuum state with $h = 0$ are in this range.

Interpretation of p as spacetime momentum: As in the linear dilaton case in the previous section, we face some interpretational issues surrounding spacetime momentum. For a state $|a\rangle = \frac{Q}{2} + ip$, we will take the imaginary part p to be the spacetime momentum in analogy to the free boson theory. We can argue this in a couple of ways. Firstly, considering the Hamiltonian $H_0 = L_0 + \bar{L}_0 - c/12$ with the eigenvalue of L_0 being $h = a(Q-a)$ in the semi-classical Liouville theory (see [74, p.13]), we find that for $a \in \frac{Q}{2} + ip$, $p \in \mathbb{R}$ and $c = 1 + 6Q^2$

$$H_0 = \frac{Q^2}{2} + 2p^2 - \frac{1 + 6Q^2}{12} = 2p^2 - \frac{1}{12}, \tag{5.46}$$

which is simply the energy with an asymptotic momentum p.

Secondly, in the vertex operators language, V_a is reflected off the Liouville potential (5.37). Since for $a \in \frac{Q}{2} + ip$, $p \in \mathbb{R}$, we have $Q - a = \frac{Q}{2} - ip$, the flipping of the sign on reflection $p \to -p$ indicates that p indeed is the physical momentum of the theory.

Two-point function: Due to the Liouville potential, the two-point function is given by

$$G_{\text{sl.}}(a_2, a_1) = 2\pi\delta(a_2 - Q + a_1) + R_{\text{sl.}}(a_1)\delta(a_2 - a_1). \tag{5.47}$$

The first term follows from the fact that $Q - a_1$ labels the operator conjugate to V_{a_1}. In the second term, we can see that since p_1 in $a_1 = \frac{Q}{2} + ip_1$ is the physical momentum, $V_{Q/2-ip_1}$ is the operator reflected off the Liouville potential which leads to $\delta(a_2 - (Q/2 - ip_1)^*) = \delta(a_2 - a_1)$. We will comment a little more on the physical interpretation of Im a as momentum below.

5.2.2. The $\mathcal{N} = 1$ Liouville Theory

The $\mathcal{N} = 1$ Liouville theory is defined via the following action [68]

$$S = \int d^2z d\theta d\bar{\theta} \left(\frac{1}{2\pi} D\Phi\bar{D}\Phi + 2i\mu e^{b\Phi} \right), \tag{5.48}$$

with central charge given by [73]

$$c = \frac{3}{2}(1 + 2Q^2). \tag{5.49}$$

The superfield Φ is given by the expansion $\Phi = \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + i\theta\bar{\theta}F$ and the supercovariant derivatives are defined via $D = \partial_\theta + \theta\partial_z$ and $\bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$. The NS sector vertex operators are a simple extension of the $\mathcal{N} = 0$ case

$$V_a^{\text{NS}} = e^{a\phi}, \tag{5.50}$$

with conformal dimensions

$$h_a^{\text{NS}} = \frac{1}{2}a(Q - a). \tag{5.51}$$

These correspond to the spacetime bosons. The R sector vertex operators that correspond to the spacetime fermions require introducing extra spin fields $\sigma^\pm$

$$V_a^{\text{R},\pm} = \sigma^\pm e^{a\phi}. \tag{5.52}$$

These have conformal dimensions given by [73]

$$h_a^{\mathrm{R}} = \frac{1}{2}a(Q-a) + \frac{1}{16}. \tag{5.53}$$

Three-point functions: The correlators of 'ordinary' (spacelike) $\mathcal{N}=1$ Liouville theory are given by (see Fedenhagen and Wellig [73] and Fukuda and Hosomichi [76])

$$C_{\mathrm{sl.}}^{\mathrm{NS}} = A_{\mathrm{sl.}}^{\mathrm{NS}} b^{\hat{Q}(Q-\hat{a})} \frac{\Upsilon'_{\mathrm{NS}}(0)}{\Upsilon_b(\frac{\hat{a}}{2})\Upsilon_b(\frac{\hat{a}-Q}{2})} \times \prod_{j=1}^{3} \frac{\Upsilon_b(a_j)\Upsilon_b(a_j+\frac{Q}{2})}{\Upsilon_b(\frac{\hat{a}_j}{2})\Upsilon_b(\frac{\hat{a}_j+Q}{2})} \tag{5.54}$$

where $Q = b + b^{-1}$, $\hat{Q} = b^{-1} - b$, $\hat{a} = \sum_i a_i$, $\hat{a}_i = \hat{a} - 2a_i$ and

$$\Upsilon_{\mathrm{NS}}(x) = \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right). \tag{5.55}$$

The term $\Upsilon'_{\mathrm{NS}}(0)$ will be dealt with later in the section covering the analytic continuation of this theory. As in the bosonic case, the theory possesses a duality $b \to b^{-1}$ and hence there is an analogous expression for b^{-1}.

5.2.3. The $\mathcal{N}=2$ Liouville Theory

In our ϵ-Gepner models, $\mathcal{N}=1$ Liouville may not provide an ϵ that is small enough to suit perturbative theory to remain valid. The $\mathcal{N}=2$ theory however does provide a small central charge and hence is more suitable to model building for us. In contrast to the $\mathcal{N}=1$ case, the $\mathcal{N}=2$ extension of Liouville theory is much less trivial. Here we review the $\mathcal{N}=2$ theory with the conventions of Hosomichi [77]. Since this is the case we are mainly interested in, we review it in a bit more detail. In this section, we also explicitly compute modular invariance properties of the $\mathcal{N}=2$ partition function.

The Action: The action of $\mathcal{N}=2$ Liouville theory on a flat Euclidean worldsheet may be written as

$$S_L = \frac{1}{8\pi}\int d^2z d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \Phi\bar{\Phi} + \frac{\mu}{2\pi}\int d^2z d\theta^+ d\theta^- e^{-\sqrt{k/2}\,\Phi} \tag{5.56}$$

$$+ \frac{\bar{\mu}}{2\pi}\int d^2z d\theta^+ d\theta^- e^{-\sqrt{k/2}\,\bar{\Phi}}, \tag{5.57}$$

where the chiral superfield Φ

$$\Phi = \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\bar{\theta}^+\bar{\psi}_+ + 2\theta^+\bar{\theta}^+F + \dots \tag{5.58}$$

$$\bar{\Phi} = \bar{\phi} + i\sqrt{2}\theta^-\psi_- + i\sqrt{2}\bar{\theta}^-\bar{\psi}_- + 2\theta^-\bar{\theta}^-\bar{F} + \dots, \tag{5.59}$$

satisfies

$$(\partial/\partial\theta^- - i\theta^+\partial)\Phi = (\partial/\partial\bar{\theta}^- - i\bar{\theta}^+\bar{\partial})\Phi = 0, \tag{5.60}$$

$$(\partial/\partial\theta^+ - i\theta^-\partial)\bar{\Phi} = (\partial/\partial\bar{\theta}^+ - i\bar{\theta}^-\bar{\partial})\bar{\Phi} = 0. \tag{5.61}$$

This leads to an $\mathcal{N} = 2$ superconformal algebra with *central charge*

$$c = \frac{3(k+2)}{k}. \tag{5.62}$$

Vertex Operators: In [77], the *bulk* vertex operators of the following form are considered

$$V^{j,(s,\bar{s})}_{m,\bar{m}} = e^{\sqrt{k/2}\{j(\rho_L+\rho_R)+i(m+s)\theta_L+i(\bar{m}+\bar{s})\theta_R\}+isH_L+i\bar{s}H_R}, \tag{5.63}$$

where the $\phi_{L,R} = \rho_{L,R} + i\theta_{L,R}$ and the fermions $\psi_\pm, \bar{\psi}_\pm$ are bosonised in terms of the compact boson $H = H_L + H_R$

$$\psi_\pm = \sqrt{2}e^{\pm iH_L} \text{ and } \psi_\pm = \sqrt{2}e^{\pm iH_L}. \tag{5.64}$$

The vertex operator (5.63) is parameterised by j, s, m along with their right-moving counterparts. The parameters s and $\bar{s}$) determine the monodromy of the fermions $\psi_\pm, \bar{\psi}_\pm$ around V giving the two sectors in the supersymmetric theories: the NS-sector with $s, \bar{s} \in \mathbb{Z}$ and the R-sector with $s, \bar{s} \in \mathbb{Z} + \frac{1}{2}$. The interaction terms must be single-valued around the vertex operators V in order for their correlators to be calculated perturbatively. This lead to a constraint $m - \bar{m} \in \mathbb{Z}$ on the parameters. These are the so-called *perturbatively well-behaved* operators. Here, we will ignore all other operators where $m, \bar{m} \notin \mathbb{Z}$. To understand the role of $m, \bar{m}$ (initially considering the case where $s = \bar{s} = 0$), we look at the behaviour of θ, which is the phase of the chiral field $\exp(-\sqrt{k/2}\Phi)$, around an operator V with $m-\bar{m}\in\mathbb{Z}$

$$\theta(z)V^j_{m,\bar{m}}(0)\sim -i\sqrt{2/k}(m\ln z+\bar{m}\ln\bar{z})V^j_{m,\bar{m}}(0), \tag{5.65}$$

i.e., θ has a periodicity $2\pi\sqrt{2/k})$ and $m-\bar{m}$ corresponds to the winding number along the θ-direction. Correspondingly, the momentum in the θ-direction $(\sqrt{2k})^{-1}(m+\bar{m})$ should be quantised in units of $\sqrt{k/2}$. This quantisation requirement imposes a further condition on $m,\bar{m}$,

$$m=(kn+w)/2,\ \ \bar{m}=(kn-w)/2,\ \ n,w\in\mathbb{Z}. \tag{5.66}$$

A similar argument for operator with non-zero $s,\bar{s}$ leads to the quantisation law

$$m-\bar{m}\in\mathbb{Z}, m+s+\bar{m}+\bar{s}\in k\mathbb{Z}. \tag{5.67}$$

The $\mathcal{N}=2$ Liouville theory satisfies the usual $\mathcal{N}=2$ superconformal algebra generated by currents $T,G^\pm$ and J, commutation relations of whose modes are given above. The left-moving vertex operator

$$V^{j,s}_m=e^{\sqrt{(k/2)}\{j\rho_L+i(m+s)\theta_L\}+isH_L} \tag{5.68}$$

corresponds to a state $|j,m,s\rangle$ with L_0,J_0 eigenvalues h,Q as follows

$$h=\frac{(m+s)-j(j+1)}{k}-\frac{s^2}{2}, \tag{5.69}$$

$$Q=\frac{2(m+s)}{k}+s. \tag{5.70}$$

The right-moving state $|j,\bar{m},\bar{s}\rangle$ has a similar expression. This state is annihilated by $G^\pm_{r\geq -s\mp\frac{1}{2}}$. The vertex operators with $s=0$ correspond to the primary states of the NS-sector and s represents the amount of spectral flow. That is, for an arbitrary state

$$|j,m,s\rangle=U^s|j,m,0\rangle, \tag{5.71}$$

where the spectral flow operator U^s; see Eq. 2.19 [77]. The action of the supercurrents $G^{\pm}(z)$ on these operators is as follows

$$G^{\pm}(z)V_m^{j(s)}(0) \sim -i\sqrt{2/k}\,(j \pm m)\frac{1}{z^{\mp s-1}}V_{m\mp 1}^{j(s\pm 1)}(0). \tag{5.72}$$

It follows from this equation that the states with

$$j = \mp m \tag{5.73}$$

are annihilated by the action of the supercurrent modes $G^{\pm}_{\pm s-1/2}$. There are unitarity bounds on j

$$\left(-\frac{k}{2}-1\right) < j < -\frac{1}{2} \quad \text{and} \quad \pm\, m \in j - \mathbb{Z}_{\geq 0}. \tag{5.74}$$

or

$$-1 < j < 0 \quad \text{and} \quad j \in -\frac{1}{2} + i\mathbb{R}. \tag{5.75}$$

Primary representations: We consider the NS sector primary states with $s = 0$. Using (5.73), the conformal dimension (5.69) and the $U(1)$ charge (5.70) become

$$h = -\frac{j}{k} \qquad \text{and} \qquad Q = \mp\frac{2j}{k}, \tag{5.76}$$

for $j = \mp m$ respectively i.e., for chiral and anti-chiral states respectively. Our free parameter in Liouville theory is k, so we use the range for the label j given in terms of k by (5.74) to determine what chiral primaries there are. For $j = -1/2$, we get $h = -j/k = 1/(2k)$ and for $j = -k/2 - 1$, we have $h = \frac{k+2}{2k}$. We therefor have a continuous set of representations with conformal dimension

$$\frac{1}{2k} < h < \frac{k+2}{2k}. \tag{5.77}$$

Since $k = \frac{6}{c-3}$, the choice of the central charge ultimately determines the spectrum of primary fields.

Characters of Liouville Representations: Now we turn our attention to the characters of these representations. In the bulk, the character of such a continuous representation is[1]

$$\chi_m^{j,s}(\tau,\alpha) = q^{\frac{m^2}{k} - \frac{(2j+1)^2}{4k} + \frac{s^2}{2}} \, z^{\frac{2m}{k}+s} \, \frac{\vartheta_3(\alpha + s\tau, \tau)}{\eta(\tau)^3}, \tag{5.78}$$

where $q = e^{2\pi i\tau}$, $z = e^{2\pi i\alpha}$ and Dedekind's η-function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n), \tag{5.79}$$

whereas the ϑ_3 is Jacobi's 3rd theta-function, $\vartheta_3(\alpha,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} z^n$. Note that the conventions for ϑ_3 as well as for parameters q and z in terms of τ and α respectively are different from those used in the bosonic and $\mathcal{N} = 1$ theories, including the timelike extensions in the rest of this text. This is deisgned in part to keep the conventions in this section aligned with those of [77].

Liouville Partition Function: Finally, we write down the partition function for this $\mathcal{N} = 2$ Liouville theory (while setting $\alpha = 0$):

$$Z(\tau,\bar{\tau}) = \int \chi_{j,m,s}(\tau) \, \chi_{j,\bar{m},\bar{s}}(\bar{\tau}), \tag{5.80}$$

where the integration is carried out over all representation with allowed values of $j, m, \bar{m}, s, \bar{s}$. We can write this as a product of three terms

$$Z(\tau,\bar{\tau}) = Z(j;\tau,\bar{\tau}) \, Z(m,\bar{m};\tau,\bar{\tau}) \, Z(s,\bar{s};\tau,\bar{\tau}), \tag{5.81}$$

[1] The character given here is based on [77]. We check explicitly that it matches the one used by [78]

$$\chi_m^{j',0}(\tau,\alpha) = q^{\frac{m^2}{k} - \frac{j'(j'-1)}{k} - \frac{1}{4k}} \, z^{\frac{2m}{k}} \, \frac{\vartheta_3(\alpha + s\tau, \tau)}{\eta(\tau)^3}.$$

For $j, j' = -\frac{1}{2} + ix, x \in \mathbb{R}$, the corresponding j terms in χ are equal: $(2j+1)^2 = 4j'(j'-1) + 1 = -4x^2$.

where the individual parts are as follows, each simplified in order to facilitate the modular invariance calculations.

$$
\begin{aligned}
Z(j;\tau,\bar\tau) &= \int_{j\in -\frac{1}{2}+i\mathbb{R}} e^{-2\pi i(\tau-\bar\tau)(2j+1)^2/4k} dj \\
&= i\int_{x\in\mathbb{R}} e^{2\pi i(\tau-\bar\tau)x^2/k} dx \\
&= i\left[-\frac{\pi k}{2\pi i(\tau-\bar\tau)}\right]^{\frac{1}{2}} = i\left[\frac{k}{4\mathrm{Im}\,\tau}\right]^{\frac{1}{2}}.
\end{aligned} \tag{5.82}
$$

Here, in the second line we have used $j = -\frac{1}{2} + ix, x \in \mathbb{R}$ to obtain $(2j+1)^2 = -4x^2, dj = idx$, while in the third line, we observe that for $a = -2\pi i(\tau - \bar\tau)/k$, this is just the Gaussian integral $\int e^{-ax}dx = \sqrt{\pi/a}$. Next is the $m, \bar m$ part where the various quantisation conditions restrict the domain of integration

$$
Z(m,\bar m;\tau,\bar\tau) = \int\limits_{\substack{m-\bar m\in\mathbb{Z} \\ m,\bar m=(kn\pm w)/2 \\ n,w\in\mathbb{Z}}} e^{2\pi i(\tau m^2-\bar\tau\bar m^2)/k} dm d\bar m. \tag{5.83}
$$

By setting $\alpha = 0$, we can write the $s, \bar s$ part of Z as a sum over integers or half-integers

$$
\begin{aligned}
Z(s,\bar s;\tau,\bar\tau) &= \sum_{s,\bar s\in\mathbb{Z} \text{ or } s,\bar s\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}s^2}\bar q^{\frac{1}{2}\bar s^2}\frac{\vartheta_3(s\tau,\tau)}{\eta(\tau)^3}\frac{\vartheta_3(s\bar\tau,\bar\tau)}{\bar\eta(\bar\tau)^3} \\
&= \frac{1}{\eta(\tau)^3\bar\eta(\bar\tau)^3}\sum_{s,\bar s\in\mathbb{Z} \text{ or } s,\bar s\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}s^2}\bar q^{\frac{1}{2}\bar s^2}\Big(\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}e^{2\pi i s\tau n}\Big)\Big(\sum_{l\in\mathbb{Z}} q^{\frac{1}{2}l^2}e^{2\pi i\bar s\bar\tau l}\Big) \\
&= \frac{1}{\eta(\tau)^3\bar\eta(\bar\tau)^3}\sum_{s,\bar s\in\mathbb{Z} \text{ or } s,\bar s\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}s^2}\bar q^{\frac{1}{2}\bar s^2}\Big(\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+s)^2-\frac{1}{2}s^2}\Big)\Big(\sum_{l\in\mathbb{Z}} q^{\frac{1}{2}(l+\bar s)^2-\frac{1}{2}\bar s^2}\Big) \\
&= \frac{1}{\eta(\tau)^3\bar\eta(\bar\tau)^3}\sum_{s,\bar s\in\mathbb{Z} \text{ or } s,\bar s\in\mathbb{Z}+\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+s)^2}\sum_{l\in\mathbb{Z}}\bar q^{\frac{1}{2}(l+\bar s)^2}.
\end{aligned}
$$

In the case of the NS sector ($s, \bar s \in \mathbb{Z}$), it further simplifies to

$$
Z_{\mathrm{NS}}(s,\bar s;\tau,\bar\tau) = \frac{1}{\eta(\tau)^3\bar\eta(\bar\tau)^3}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\sum_{l\in\mathbb{Z}}\bar q^{\frac{1}{2}l^2} = \frac{\vartheta_3(0,\tau)\vartheta_3(0,\bar\tau)}{\eta(\tau)^3\bar\eta(\bar\tau)^3}. \tag{5.84}
$$

The Ramond sector stays as before

$$Z_{\mathrm{R}}(s,\bar{s};\tau,\bar{\tau}) = \frac{1}{\eta(\tau)^3\bar{\eta}(\bar{\tau})^3}\sum_{s,\bar{s}\in\mathbb{Z}+\frac{1}{2}}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+s)^2}\sum_{l\in\mathbb{Z}}\bar{q}^{\frac{1}{2}(l+\bar{s})^2}. \tag{5.85}$$

We demonstrate modular invariance of this partition function by considering how the individual terms transform under the modular transformations $T:\tau\to\tau+1$ and $S:\tau\to-1/\tau$. We begin with (5.82),

$$\begin{aligned} Z(j;-1/\tau,-1/\bar{\tau}) &= i\int_{x\in\mathbb{R}} e^{-2\pi i(\frac{1}{\tau}-\frac{1}{\bar{\tau}})x^2/k}dx \\ &= i\left[\frac{\pi k}{2\pi i(1/\tau-1/\bar{\tau})}\right]^{\frac{1}{2}} = i\left[\frac{k\tau\bar{\tau}}{4\mathrm{Im}\ \tau}\right]^{\frac{1}{2}} \\ &= \sqrt{\tau\bar{\tau}}\ Z(j;\tau,\bar{\tau}). \end{aligned}$$

Modular invariance of $Z(m,\bar{m};\tau,\bar{\tau})$ may be analysed by comparison with a free boson compactified on a circle of radius r. Our quantisation conditions (5.66) are analogous to those of the left and right moving charges $g,\bar{g}$ of the compactified free boson

$$g = \frac{p}{2r}+rw \ \text{ and } \ \bar{g} = \frac{p}{2r}-rw \quad \text{for } p,w\in\mathbb{Z}. \tag{5.86}$$

For $k=1/(2r^2)$, we can rescale $\tau_{\text{Liouville}} = 2r^2\tau_{\text{free boson}}$ in the partition function

$$q^{\frac{1}{2}g^2} = e^{2\pi i\tau\cdot\frac{1}{2}(\frac{n}{2r}+rw)^2} = e^{2\pi i\tau\cdot\frac{1}{2}r^2(\frac{n}{2r^2}+w)^2} = e^{2\pi i\tau\cdot 2r^2\left(\frac{\frac{n}{2r^2}+rw}{2}\right)^2}. \tag{5.87}$$

Since, the partition function of the compactified free boson

$$Z_{\mathrm{FB}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{p,w\in\mathbb{Z}} q^{\frac{1}{2}g^2}\bar{q}^{\frac{1}{2}\bar{g}^2}, \tag{5.88}$$

is modular invariant, $Z(m,\bar{m};\tau,\bar{\tau})$ is modular invariant for the choice of compactification radius $r=\sqrt{\frac{1}{2k}}$.

Finally, we turn to the $Z(s,\bar{s};\tau,\bar{\tau})$ part. In the NS sector, we first note the effect of modular

S transformation on functions ϑ_3 and η

$$\vartheta_3(0,-1/\tau) = \sqrt{-i\tau}\;\vartheta_3(0,\tau) \quad \text{and} \quad \eta(-1/\tau) = \sqrt{-i\tau}\;\eta(\tau),$$

and similarly for terms with $\bar{\tau}$. Hence

$$\begin{aligned} Z_{\rm NS}(s,\bar{s};-1/\tau,-1/\bar{\tau}) &= -\frac{\vartheta_3(0,\tau)\vartheta_3(0,\bar{\tau})}{\tau\bar{\tau}\eta(\tau)^3\bar{\eta}(\bar{\tau})^3} \\ &= -\frac{1}{\tau\bar{\tau}}\; Z_{\rm NS}(s,\bar{s};\tau,\bar{\tau}). \end{aligned} \tag{5.89}$$

Collecting the transformations of all three terms in (5.81) under S in the NS sector, we get

$$Z_{\rm NS}(-1/\tau,-1/\bar{\tau}) = -\frac{1}{\sqrt{\tau\bar{\tau}}}\; Z_{\rm NS}(\tau,\bar{\tau}). \tag{5.90}$$

We have not explicitly checked the modular invariance of the R sector at this stage.

5.2.4. The Boundary Theory – D-branes in Liouville Theory

Since any noncompact theory with continuous degrees of freedom may be a candidate for the external sector of our ϵ-Gepner models, Liouville theory despite the lack of its cosmological interpretation in the effective physics picture described in Chapter 4, may be considered. To that end, we would like to have spacetime filling branes in the theory. Indeed, the Neumann conditions in the boundary Liouville theory were studied by [88] and [89] and the branes are known as FZZT branes. We will not review them here.

Chapter 6

Timelike Liouville Theory

During the initial stages of this work, understanding time-dependent backgrounds was a key objective. To this end, we have attempted to develop a better understanding of the timelike Liouville theory with supersymmetry. In particular, we studied McElgin's recipe for defining timelike Liouville theory. This section documents the body of work, starting with a brief review of Liouville theory and leading up to some results obtained for the timelike $\mathcal{N} = 1$ Liouville theory.

We would like to state from the outset that the results presented here are negative in that McElgin's approach for the bosonic case fails to extend to the supersymmetric case. They nonetheless lay the foundation for future work by the author to explore further techniques. It should be noted that the recent work of Harlow, Maltz and Witten [79] also deals with analytic continuation of Liouville theory using path integral methods. In particular, they make some observations on the timelike bosonic Liouville theory, which may cast new light on the problem but fall short of leading to a full definition of the timelike theory.

6.1. Timelike Bosonic Liouville Theory

The $p \to \infty$ limit of the unitary minimal models labelled by $(p, p+1)$ – a $c = 1$ nonrational CFT – was shown to bear some resemblance to Liouville theory in [69]. It was further shown by Schomerus [70] that such a theory may be obtained by analytically continuing the Liouville theory to $c = 1$ and a discussion of the timelike regime of $c < 1$ was also given. Before we describe our attempts to find a supersymmetric timelike theory with $\mathcal{N} = 1$, we shall look at the bosonic version that we aim to extend. This section is based on an approach taken by McElgin [72] that studies the models instigated by [69] and [70] in more detail for the $c < 1$ branch.

Timelike Liouville ($c \leq 1$): The analytic continuation of ordinary Liouville theory leads to a timelike Liouville theory parameterised by $\beta = ib \in \mathbb{R}$. The central charge may be written as

$$c = 1 - 6\Lambda^2, \quad \text{where} \quad \Lambda = -iQ = \beta^{-1} - \beta \in \mathbb{R}. \tag{6.1}$$

In this theory, $b \in i\mathbb{R}$ and $c \leq 1$. The vertex operators in the timelike theory are labelled by $\alpha = \frac{1}{2}(-\Lambda - i\omega)$ where $\omega = -iP$ in terms of $P \in \mathbb{R}$ above. These have conformal dimensions $h_\alpha = \alpha(\Lambda + \alpha)$. Arguments similar to above show that

$$h_\alpha = -\frac{\Lambda^2}{4} - \omega^2 > -\frac{\Lambda^2}{4}, \tag{6.2}$$

since $\omega \in i\mathbb{R}$. In terms of the central charge, the bounds on the conformal dimensions of the primaries in the spacelike and timelike theories become $h_a > \frac{c-1}{24}$ and $h_\alpha < \frac{c-1}{24}$ respectively.

In analogy to the ordinary spacelike Liouville theory, $V_{-\Lambda-a} = S(a)V_a$ holds for reflected vertex operators, where now $a = -\Lambda/2 - i\omega$, $\omega \in \mathbb{R}$ allowing the identification of ω with the physical momentum. Note that since this is the timelike theory, this ω is the 0^{th} component of the momentum vector and hence can be seen as the energy.

Continuation of the spacelike theory: A straightforward analytic continuation of $C_{\text{sl.}}$ to $b \in i\mathbb{R}$ does not work since $C_{\text{sl.}}$ is wildly discontinuous at $b \in i\mathbb{R}$. Moreover, for $b \in i\mathbb{R}$, solution to the analytically continued shift relation (5.30) is not quite unique; see [71] for details. The crux of the recipe outlined in [72] is that a unique solution can nonetheless be found for the shift relations (5.30) for a real $\beta = ib$ and for $\alpha_j = ia_j$. Essentially the following 'continued' shift relation is found

$$\frac{C_{\text{tl.}}(\alpha_3, \alpha_2, \alpha_1 + \beta|\beta)}{C_{\text{tl.}}(\alpha_3, \alpha_2, \alpha_1|\beta)} = H_{\text{tl.}}(\alpha_3, \alpha_2, \alpha_1|\beta), \tag{6.3}$$

where $H_{\text{tl.}}(\alpha_3, \alpha_2, \alpha_1|\beta) = H_{\text{sl.}}(-i\alpha_3, -i\alpha_2, -i\alpha_1| - i\beta)$. When combined with the analogous expression for $\beta \to \beta^{-1}$, the 'candidate' three-point correlator of this continued theory is given by

$$C_{\text{tl.}}(\alpha_3, \alpha_2, \alpha_1|\beta) = A_{\text{tl.}} \times \left(\beta^{2(\beta^{-1}+\beta)}\right)^{(-\Lambda-\hat{\alpha})} \times \frac{\Upsilon_\beta(\beta - \hat{\alpha} - \Lambda)}{\Upsilon_\beta(\beta)} \prod_j \frac{\Upsilon_\beta(\beta - \hat{\alpha}_j)}{\Upsilon_b(\beta - 2\alpha_j)}, \tag{6.4}$$

where $\hat{\alpha} = \sum_j \alpha_j$ and $\hat{\alpha}_j = \hat{\alpha} - 2\alpha_j$. The analogous two-point function derived in a way similar to the spacelike case would then be given by

$$\begin{aligned} G_{\text{tl.}}(\alpha_2, \alpha_1) = C_{\text{tl.}}(0, \alpha_2, \alpha_1) = A_{\text{tl.}} \times \left(\beta^{2(\beta^{-1}-\beta)}\right)^{p_+} \\ \times \frac{\Upsilon_\beta(\beta + p_+)\Upsilon_\beta(\beta - p_+)\Upsilon_\beta(\beta + p_-)\Upsilon_\beta(\beta - p_-)}{\Upsilon_\beta^2(\beta)\Upsilon_\beta(\beta - p_1)\Upsilon(\beta - p_2)}. \end{aligned} \tag{6.5}$$

where $p_\pm = (p_1 \pm p_2)/2$. Since this does not contain a delta function, it is not diagonal in the α labels and hence does not lead to a good scalar product on the space of primary fields. Note that just like $C_{\text{sl.}}$ was discontinuous for $b \to i\mathbb{R}$, $C_{\text{tl.}}$ is discontinuous for $b \to \mathbb{R}$. Since the candidate expression for $C_{\text{tl.}}$ in (6.3) does not provide a way to define a unitary conformal field theory for charges $\{\alpha_j\}$, McElgin is forced to explore different techniques for computing the three-point function for the timelike theory, these do lead to a two-point function but only in a special case when the theory corresponds to minimal models with labels $(1, q)$. Since the work here aims to extend McElgin's bosonic recipe to the $\mathcal{N} = 1$ case in section 6.2, we will briefly review it next.

McElgin's procedure involves analysing the ratio of $C_{\text{sl.}}$ to $C_{\text{tl.}}$ as it must be related by a doubly periodic function in each of the charges a_j for Im $b^2 \neq 0$ [72]

$$\frac{C_{\text{sl.}}(a_3, a_2, a_1|b)}{C_{\text{tl.}}(ia_3, ia_2, ia_1|ib)} = \frac{1}{b} T(a_3, a_2, a_1|b). \tag{6.6}$$

The function T is the focus of attention in this argument. It must be doubly periodic with periods b and b^{-1}

$$T(a_3, a_2, a_1|b) = T(a_1 + b, a_2 + b, a_3 + b|b) = T\left(a_3 + \frac{1}{b}, a_2 + \frac{1}{b}, a_3 + \frac{1}{b}|b\right), \tag{6.7}$$

or equivalently

$$T\Big(\frac{a_1}{b},\frac{a_2}{b},\frac{a_3}{b}\big|b\Big)=T\Big(\frac{a_1}{b}+1,\frac{a_2}{b}+1,\frac{a_3}{b}+1\big|b\Big)=T\Big(\frac{a_2}{b}+\tau,\frac{a_2}{b}+\tau,\frac{a_3}{b}+\tau\big|b\Big), \quad (6.8)$$

Using properties of the $\Upsilon_b(z)$ function given in the appendix, this was computed by [71, 70] to be

$$T(\alpha_3,\alpha_2,\alpha_1|b)=e^{-\pi i(Q-2\hat{\alpha})/b}\frac{\vartheta_1'(0|\tau)}{\vartheta_1(\frac{\hat{\alpha}-Q}{b}|\tau)}\prod_j\frac{\vartheta_1(\frac{2\alpha_j}{b}|\tau)}{\vartheta_1(\frac{\hat{\alpha}_j}{b}|\tau)}, \quad (6.9)$$

where

$$\tau=\frac{1}{b^2}. \quad (6.10)$$

Note that ϑ_1 (a Jacobi's theta function) is defined only on the upper half-plane of τ, i.e., Im $\tau>0$ or for Im $1/\tau=$ Im $b^2<0$. It is highly singular in the limit Im $\tau\downarrow 0$. Definitions and various useful identities of Jacobi's theta functions are given in the appendix. We have explicitly verified this up to the ratio of scale factors $A_{\text{sl.}}/A_{\text{tl.}}$. Next, we will summarise some of the arguments that lead [72] to his understanding of the two-point function in the timelike case, taking into account other work done on these limits by in particular [70].

Getting a non-trivial $C_{\text{tl.}}$: Based on the analyticity properties of T in the bosonic case in the limit Im $\tau\to 0^+$, it is concluded in [72], that the only theories with

$$b\to -i\sqrt{p/q} \quad \text{or} \quad \beta\to\sqrt{p/q}. \quad (6.11)$$

lead to sensible $C_{\text{tl.}}$. T is also required to be non-trivial – for instance, not independent of a_j or at least non-constant – or else (6.9) would imply that $C_{\text{tl.}}$ alone could lead to a diagonal two-point function, which is shown in [72] not be possible. Using properties of ϑ_1 functions, in particular

$$\vartheta_1\Big(\frac{\hat{a}-Q}{b}\Big)=e^{-\pi i(Q-2\hat{a}+b)/b}\vartheta_1\Big(\frac{\hat{a}}{b}\Big) \quad (6.12)$$

and McElgin's eq. (134) which reads

$$\vartheta_1(2x)\prod_j \vartheta_1(2x_j) = \vartheta_1(\hat{x}-x)\prod_j \vartheta_1(\hat{x}-2x_j+x) - \vartheta_1(\hat{x}+x)\prod_j \vartheta_1(\hat{x}-2x_j-x) \tag{6.13}$$

we can derive an expression that converts the product of quotients of ϑ_1 into a sum of their quotients as follows

$$-\frac{\vartheta_1'(0|\tau)}{\vartheta_1(\hat{x}|\tau)}\prod_j \frac{\vartheta_1(2x_j|\tau)}{\vartheta_1(\hat{x}_j|\tau)} = \frac{\vartheta_1'(\hat{x}|\tau)}{\vartheta_1(\hat{x}|\tau)} - \sum_j \frac{\vartheta_1'(\hat{x}_j|\tau)}{\vartheta_1(\hat{x}_j|\tau)}, \tag{6.14}$$

This means that we can write T as

$$\begin{aligned} T &= -\frac{\vartheta_1'(0|\tau)}{\vartheta_1(\frac{\hat{a}}{b}|\tau)}\prod_j \frac{\vartheta_1(\frac{2a_j}{b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{b}|\tau)} = \frac{\vartheta_1'(\frac{\hat{a}}{b}|\tau)}{\vartheta_1(\frac{\hat{a}}{b}|\tau)} - \sum_j \frac{\vartheta_1'(\frac{\hat{a}_j}{b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{b}|\tau)} \\ &= \frac{d}{dx}\ln\left(\frac{\vartheta_1(\frac{x}{b}|\tau)}{\prod_j \vartheta_1(\frac{x_j}{b}|\tau)}\right)\Bigg|_{x=\hat{a},x_j=\hat{a}_j}. \end{aligned} \tag{6.15}$$

Although ϑ_1 is only quasi-periodic, the complete function T is doubly periodic (6.7),(6.8). Hence in the limit Im $\tau{\downarrow}0$, when $\tau \to r \in \mathbb{R}$, we can use this fact to determine what additional restrictions must be placed on r in order to have a non-trivial T. For $r \notin \mathbb{Q}$, T in (6.15) would have to be constant as the periodicities of T would not have a rational ratio for Im $\tau{\downarrow}0$. Therefore, we must have $r \in \mathbb{Q}$ for a non-trivial function T consistent with the fact $C_{\text{tl.}}$ must be supplemented with another function of the charges a_i in order to produce a diagonalisable set of states in the Hilbert space. So we set $b = -i\sqrt{p/q}$ for two coprime natural numbers p, q.

Obtaining a diagonal two-point function: Furthermore, the procedure in [72] only leads to sensible 2-point functions when $p = 1$, i.e.

$$b = -i\sqrt{1/q} \quad \text{or} \quad \beta = \sqrt{1/q}. \tag{6.16}$$

This may be seen as follows. Let $\tau = -r + i\epsilon$, where $r \in \mathbb{Q}$. The Liouville vertex operator labels α, also referred to as 'charges' usually take their values in the range $\frac{Q}{2}+i\mathbb{R}$. In McElgin's conventions, each $2\alpha \in Q+i\mathbb{R}$. So in the limit $b \to -i\sqrt{p/q}$, the parameters of the ϑ_1 functions in T become

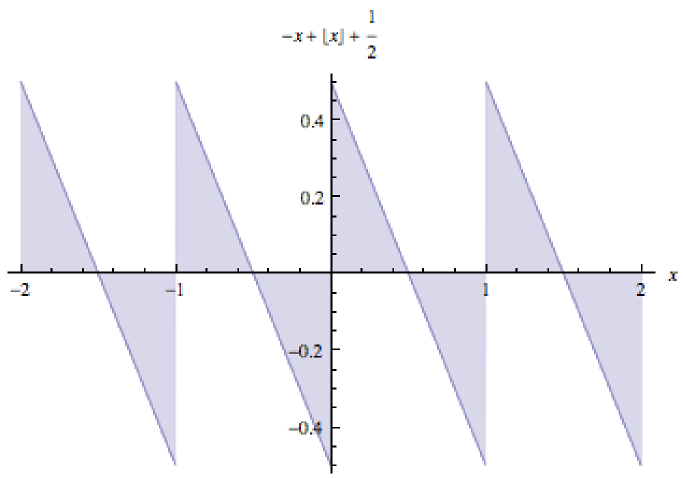
$$2x_j = 2\hat{a}_j/b = 1 - q/p - k_j\sqrt{p/q} \tag{6.17}$$

McElgin claims that the only case where a diagonal two-point function exists is that of $k_j \in \mathbb{R}$. The $k_j \in i\mathbb{R}$ case has been left without conclusion in [72] and has not been resolved yet by us. This will partially be the subject of future work.

Understanding the ratio $\vartheta_1'(z|\tau)/\vartheta_1(z|\tau)$ is crucial to analysing the function T. McElgin's final construction of the bosonic 2-point function relies on the following periodic sawtooth function

$$\mathcal{D}_1(x) = \lim_{\epsilon\to 0} \frac{\epsilon}{2\pi}\frac{\vartheta_1'(z|\tau)}{\vartheta_1(z|\tau)} = \frac{1}{2} - (x - \lfloor x \rfloor), \qquad \tau = 1 + i\epsilon. \tag{6.18}$$

This is extended to the case of general p via



$$\mathcal{D}_p(x) \equiv \lim_{\epsilon\to 0} \frac{\epsilon}{2\pi}\frac{\vartheta_1'(x| - q/p + i\epsilon)}{\vartheta(x| - q/p + i\epsilon)} = \frac{1/2 - px + \lfloor px \rfloor}{p}, \tag{6.19}$$

where $\tau = -q/p + i\epsilon$ and $\lfloor x \rfloor$ is the largest integer less than $x \in \mathbb{R}$. This expression is independent of q. A key use of this function is that it paves the way for the Dirac δ function

$$\frac{\partial}{\partial x}\mathcal{D}_p(x) = -1 + \sum_{n\in\mathbb{Z}} \delta(px - n). \tag{6.20}$$

The δ function lets us construct a two-point function which is diagonal. For a discussion of equation (6.18), see the appendix of [70], where it is also claimed that the same sawtooth function arises as an analogous limit of ϑ_3'/ϑ_3,

$$\mathcal{D}_1(x) = \lim_{\epsilon\to 0}\frac{\epsilon}{2\pi}\frac{\vartheta_3'(z|\tau)}{\vartheta_3(z|\tau)} = \frac{1}{2} - (x - [x]). \tag{6.21}$$

This will be useful to us later in the $\mathcal{N}=1$ case. For $x_j \in \mathbb{R}$ (i.e., when $k_j \in \mathbb{R}$ in (6.17)) and using $\lfloor -x\rfloor = -\lfloor x\rfloor - 1$, McElgin finds the following expression for T

$$\lim_{\epsilon\to 0}\frac{\epsilon}{2\pi}T = p^{-1}\Big(1 + (p-q) - \frac{1}{2}(\hat{k}\sqrt{qp}\ - p + q) + \frac{1}{2}\sum_j\{(\hat{k}_j\sqrt{qp} - p + q)\}\Big) \tag{6.22}$$

where $\hat{k} = \sum_j k_k$ and $\hat{k}_j = \hat{k} - 2k_j$ in line with our previous notation. The limit (6.22) is somewhat subtle; it is taken such that $A_{\text{sl.}}/\epsilon$ is finite – see equation 147 of [72]. It is at this point that a connection is made between this approach and the model of [70] as in the case when $p = q = 1$, this non-analytic coefficient reduces to the analogous coefficient in [70]. This further corresponds to the models constructed in [69] as $c \to 1$ limit of minimal models.

Firstly, a three-point function is derived from the the ratio (6.6) as follows

$$\begin{aligned} C_{(p,q)}(\alpha_3,\alpha_2,\alpha_1) &= \lim_{\epsilon\to 0} C_{\text{sl.}}(-i\alpha_3,-i\alpha_2,-i\alpha_1|-i\beta) \\ &= 2\pi i p^{-1}\sqrt{q/p}\,\frac{A_{\text{sl.}}}{\epsilon}\,A_{\text{tl.}}^{-1}C_{\text{tl.}}(\alpha_3,\alpha_2,\alpha_1|\beta)\times\lim_{\epsilon\to 0}\frac{\epsilon}{2\pi}T. \end{aligned} \tag{6.23}$$

We have not checked explicitly that this function satisfies the shift relations of the timelike Liouville theory and this issue was not discussed in [72]. The two-point function obtained from this three-point function in the limit $\beta = ib \to \sqrt{p/q}$ will be denoted by $G_{p,q}$ and will provide a metric on the fields V_{α_j} in the limit Im $\tau\downarrow 0$ of the form

$$G_{(p,q)}(\alpha_2,\alpha_1) = \frac{1}{\epsilon}4\pi i\,\frac{p}{q}\,\frac{A_{\text{sl.}}}{A_{\text{tl.}}}\,G_{\text{tl.}}(\alpha_2,\alpha_1)\sum_{n\in\mathbb{Z}}\big\{\delta(k_+\sqrt{qp}-n) - \delta(k_-\sqrt{qp}-n)\big\}. \tag{6.24}$$

Here $k_\pm = (k_1 \pm k_2)/2$. Note that the $k_j \in \mathbb{R}$ is indeed the case we are interested in based on Teschner's arguments about unitarity of Liouville theory. For general p, q, the periodic δ functions in this expression mean that the fields of different conformal dimension will have a non-zero inner product and the theory will not be Mobius invariant. For $p = 1$ however, using results for $G_{\text{tl.}}(\alpha_2, \alpha_1)$, McElgin is able to write the two-point function as

$$G_{(1,q)}(\alpha_2, \alpha_1) = \frac{1}{\epsilon} 2i\sqrt{q} A_{\text{sl.}} \big\{ 2\pi\delta(k_+) + R_q(\alpha_1) 2\pi\delta(k_-) \big\}, \tag{6.25}$$

with the reflection coefficient $R_q(\alpha)$ given by

$$R_{(1,q)}(\alpha) = -q \frac{\gamma(-2\alpha/\sqrt{q} + 1/q)}{\gamma(2 - 2\alpha\sqrt{q} - q)}. \tag{6.26}$$

This metric is diagonal and hence, McElgin concludes that the minimal models with $p = 1$ are the only feasible theories and $p > 1$ leads to non-diagonal two-point function. Note that Harlow, Maltz and Witten [79] argue that the non-diagonality of (6.5) may simply be due to the existence of $h = 0$ operators other than the identity. It would of course be interesting to explore this claim further, in particular in light of the fact that McElgin's prescription does not carry over to the supersymmetric case.

6.2. Extension to Timelike $\mathcal{N} = 1$ Liouville Theory

Since the string models we are interested in require worldsheet supersymmetry, we have considered an extension of the arguments above to the $\mathcal{N} = 1$ Liouville case. This work will be described in this section and is in essence an extension of the procedure devised by McElgin [72] as reviewed in the previous section. We will remind the reader that at the time of writing of this thesis, the $\mathcal{N} = 1$ case is still a work in progress in that a mirror application of the procedure in the bosonic case does not lead to an analogous restriction of the $\mathcal{N} = 1$ Liouville parameter b via conditions on the two-point function. One could however adopt other approaches to explore the $\mathcal{N} = 1$ shift relations. Such approaches will be the subject of future work by the author. Although not directly fruitful, this work has nonetheless been greatly instructive for the author and techniques learnt here are proving vital for exploring other avenues.

The steps involved in the procedure are: *(a)* compute the shift relations for the spacelike theory to get the ratio $H^{\text{NS}}_{\text{sl.}}$, *(b)* analytically continue $H^{\text{NS}}_{\text{sl.}}$ to 'define' $H^{\text{NS}}_{\text{tl.}}$, *(c)* construct $C^{\text{NS}}_{\text{tl.}}$ that reproduces $H^{\text{NS}}_{\text{tl.}}$ through its shift ratio, *(d)* compute the function T^{NS} the supersymmetric version of the doubly periodic T above, *(e)* understand the analyticity properties of T^{NS}, and finally *(f)* see if a diagonal two-point function for the $\mathcal{N}=1$ theory may be defined based on the behaviour of T^{NS}.

In the following, we will largely present results of computations leaving out some of the more involved details. Proofs of two theta function identities used in the bosonic and supersymmetric cases are given in the appendices A.1.2 and A.1.3.

(a) The shift relations for the spacelike $\mathcal{N}=1$ theory: We start by computing the $\mathcal{N}=1$ analogue of the shift relations (5.30)

$$\frac{C_{\text{sl.}}^{\mathrm{NS}}(a_3,a_2,a_1+2b|b)}{C_{\text{sl.}}^{\mathrm{NS}}(a_3,a_2,a_1|b)} = H_{\text{sl.}}^{\mathrm{NS}}(a_3,a_2,a_1|b). \tag{6.27}$$

For correlators given in (5.54), this gives us

$$\begin{aligned} H_{\text{sl.}}^{\mathrm{NS}}(a_3,a_2,a_1|b) = b^{-4} &\times \frac{\gamma(a_1b)\gamma(a_1b+b^2)\gamma(\frac{2a_1b+b^2+1}{2})\gamma(\frac{2a_1b+3b^2+1}{2})}{\gamma(\frac{\hat{a}b}{2})\gamma(\frac{\hat{a}b-b^{-1}}{2})} \\ &\times \frac{\gamma(\frac{\hat{a}_1b-2b^2}{2})\gamma(\frac{\hat{a}_1b-b^2+1}{2})}{\gamma(\frac{\hat{a}_2b}{2})\gamma(\frac{\hat{a}_3b}{2})\gamma(\frac{\hat{a}_2b+b^2+1}{2})\gamma(\frac{\hat{a}_2b+b^2+1}{2})} \end{aligned} \tag{6.28}$$

Note that, following Poghosian [80], we shift the charge a_1 by $2b$ in the $\mathcal{N}=1$ theory. This is necessary in order to use the available Υ function identities. There does not seem to be a shift relation with a_1+b as in the bosonic case.

(b) Analytically continue $H_{\text{sl.}}^{\mathrm{NS}}$ to 'define' $H_{\text{tl.}}^{\mathrm{NS}}$: In the continued theory, we will need this function defined at parameters $a_j \to -i\alpha_j$ and $b \to -i\beta$

$$\begin{aligned} H_{\text{sl.}}^{\mathrm{NS}}(-i\alpha_3,-i\alpha_2,-i\alpha_1|-i\beta) = \beta^{-4}& \\ &\times \frac{\gamma(-\alpha_1b)\gamma(-\alpha_1b+b^2)\gamma(\frac{-2\alpha_1b+b^2+1}{2})\gamma(\frac{-2\alpha_1b+3b^2+1}{2})}{\gamma(-\frac{\hat{\alpha}b}{2})\gamma(\frac{-\hat{\alpha}b-b^{-1}}{2})} \\ &\times \frac{\gamma(\frac{-\hat{\alpha}_1b-2b^2}{2})\gamma(\frac{-\hat{\alpha}_1b-b^2+1}{2})}{\gamma(-\frac{\hat{\alpha}_2b}{2})\gamma(-\frac{\hat{\alpha}_3b}{2})\gamma(\frac{-\hat{\alpha}_2b+b^2+1}{2})\gamma(\frac{-\hat{\alpha}_2b+b^2+1}{2})} \\ &\equiv H_{\text{tl.}}^{\mathrm{NS}}(\alpha_3,\alpha_2,\alpha_1|\beta) \end{aligned} \tag{6.29}$$

We expect the shift relation of the continued theory to be equal to this $H_{\text{tl.}}^{\mathrm{NS}}$.

(c) Construct $C_{\text{tl.}}^{\mathrm{NS}}$ that reproduces $H_{\text{tl.}}^{\mathrm{NS}}$ through its shift ratio: There is no *a priori* way

of defining an analytically continued version of $C_{\text{sl.}}^{\text{NS}}$ (5.54). We use McElgin's definition as inspiration and use the following maps to obtain $C_{\text{tl.}}^{\text{NS}}$ from $C_{\text{sl.}}^{\text{NS}}$ above

$$\begin{aligned} b &\to \beta, & a &\to \alpha \\ Q &\to -\Lambda, & \hat{Q} &\to \hat{\Lambda} \end{aligned} \tag{6.30}$$

where $\Lambda = \beta^{-1} - \beta$ and $\hat{\Lambda} = \beta^{-1} + \beta$ now. In addition, every occurrence of $\Upsilon_b(x)$ is to be replaced by $\left\{\Upsilon_\beta(\beta - x')\right\}^{-1}$, where $x(a, Q) \mapsto x'(\alpha, \Lambda)$ according to the set of transformations (6.30). Given all these substitutions, $C_{\text{sl.}}^{\text{NS}}$ leads to the following ansatz for $C_{\text{tl.}}^{\text{NS}}$

$$\begin{aligned} C_{\text{tl.}}^{\text{NS}} = A_{\text{tl.}}^{\text{NS}} \beta^{\hat{\Lambda}(-\Lambda-\hat{\alpha})} & \frac{\Upsilon_\beta(\beta - \frac{\hat{\alpha}}{2})\Upsilon_\beta(\beta - \frac{(\hat{\alpha}+\Lambda}{2})}{Y'_{\text{NS}}(0)} \\ \times \prod_{j=1}^{3} & \frac{\Upsilon_\beta(\beta - \frac{\hat{\alpha}_j}{2})\Upsilon_\beta(\beta - \frac{(\hat{\alpha}_j-\Lambda}{2})}{\Upsilon_\beta(\beta - \alpha_j)\Upsilon_\beta(\beta - \alpha_j + \frac{\Lambda}{2})} \end{aligned} \tag{6.31}$$

Note that in this definition, Υ_{NS} of the bosonic case is replaced with Y_{NS} in the supersymmetric version. The derivative $Y'_{\text{NS}}(0)$ will be further addressed below, see (6.39). This $C_{\text{tl.}}^{\text{NS}}$ fits the ratio

$$\frac{C_{\text{tl.}}^{\text{NS}}(\alpha_3, \alpha_2, \alpha_1 + 2\beta|\beta)}{C_{\text{tl.}}^{\text{NS}}(\alpha_3, \alpha_2, \alpha_1|\beta)} = H_{\text{tl.}}^{\text{NS}}(\alpha_3, \alpha_2, \alpha_1|\beta) \tag{6.32}$$

like a glove, where $H_{\text{tl.}}^{\text{NS}}$ is given by (6.29).

(d) Compute T^{NS}: The doubly periodic function T of the bosonic case now becomes T^{NS} – the ratio of $C_{\text{sl.}}^{\text{NS}}$ and $C_{\text{tl.}}^{\text{NS}}$

$$\frac{C_{\text{sl.}}^{\text{NS}}(a_3, a_2, a_1|b)}{C_{\text{tl.}}^{\text{NS}}(ia_3, ia_2, ia_1|ib)} = b^{-1} T^{\text{NS}}(a_3, a_2, a_1|b). \tag{6.33}$$

Computing this involves making some new choices as there is a disconnect between McElgin and the expressions that appear in the $\mathcal{N} = 1$ computations. It essentially boils down to McElgin [72] using $\Upsilon_b(b)$ in the definition of $C_{\text{sl.}}$, whereas Fredenhagen and Wellig [73] use $\Upsilon'_{\text{NS}}(0)$ with

$\Upsilon_{\rm NS}$ defined as in (5.55). McElgin's definitions allow for making use of the following identity

$$\Upsilon_b(b)\Upsilon_{ib}(ib) = b^{-1}e^{\pi i\hat{Q}^2/8}e^{-\pi i\tau/4}\frac{\vartheta_1'(0|\tau)}{\vartheta_3(0|\tau)}, \tag{6.34}$$

where $\tau = b^{-2}$ as before (6.10). In the $\mathcal{N} = 1$ case, we first compute the ratio (6.33) without making any extra assumptions and get

$$\begin{aligned}\frac{C^{\rm NS}_{\rm sl.}(a_3, a_2, a_1|b)}{C^{\rm NS}_{\rm tl.}(ia_3, ia_2, ia_1|ib)} &= \frac{A^{\rm NS}_{\rm sl.}}{A^{\rm NS}_{\rm tl.}} i^{-\hat{Q}(Q-\hat{a})}\Upsilon'_{\rm NS}(0)\ Y'_{\rm NS}(0) \times e^{e^{2\pi i\tau/4}}\vartheta_3(0|\tau)^2 \\ &\times \frac{e^{\pi i(-\hat{Q}^2+4\hat{a}Q-4b^2)/8}}{\vartheta_1(\frac{\hat{a}}{2b}|\tau)\vartheta_1(\frac{\hat{a}-Q}{2b}|\tau)} \\ &\times \prod_{j=1}^{3}\frac{\vartheta_1(\frac{a_j}{b}|\tau)\vartheta_1(\frac{2a_j+Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)\vartheta_1(\frac{\hat{a}_j+Q}{2b}|\tau)}.\end{aligned} \tag{6.35}$$

Here we have used the identity

$$\Upsilon_b(x)\Upsilon_{ib}(ib - ix) = e^{\pi i(\hat{Q}+2x)^2/8}e^{-\pi i\tau/4}\frac{\vartheta_1(\frac{x}{b}|\tau)}{\vartheta_3(0|\tau)}, \tag{6.36}$$

which is valid for $\mathrm{Im}\,\tau > 0$.

In the discussion around eq. (125) of [72], McElgin states $\Upsilon_b'(0) = \Upsilon_b(b)$. In the following, the $\mathcal{N} = 1$ analogue of this will be 'constants' $\Upsilon'_{\rm NS}(0)$ and $Y'_{\rm NS}(0)$, which will be 'defined' in the same way. The spacelike case is straightforward, except that we use $2b$ as its argument as stated above

$$\Upsilon'_{\rm NS}(0) \equiv \Upsilon_{\rm NS}(2b), \tag{6.37}$$

with $\Upsilon_{\rm NS}$ defined via (5.55). The use of $2b$ instead of b is conveniently motivated by calculations to come. A justification in author's opinion is that it is also the shift parameter of the $\mathcal{N} = 1$ shift relations.

For the timelike case of the continued theory, $\Upsilon_{\rm NS}$ has to be modified before we can make use of the identities (6.34). To that end, I propose a 'continued' version of this

$$Y_{\rm NS}(x) = \Upsilon_\beta\left(\frac{x}{2}\right)\Upsilon_\beta\left(\frac{x+\Lambda}{2}\right) \tag{6.38}$$

where $\Lambda = \beta^{-1} - \beta$ as above. This is justified as Λ plays the role of Q in the continued theory. The corresponding term in the correlators is now defined as

$$Y'_{\mathrm{NS}}(0) \equiv Y_{\mathrm{NS}}(2\beta). \tag{6.39}$$

Now we attempt to simplify (6.35). First, let's take the two terms we have been looking at.

$$\begin{aligned}
&\Upsilon_{\mathrm{NS}}(2b)Y_{\mathrm{NS}}(2bi) \\
&\qquad = \Upsilon_b(b)\Upsilon_b\left(b+\frac{Q}{2}\right)\Upsilon_{ib}(ib)\Upsilon_{ib}\left(ib-\frac{iQ}{2}\right) \\
&\qquad = \gamma\left(\frac{Qb}{2}\right)b^{1-Qb}\ \Upsilon_b(b)\Upsilon_{ib}(ib)\ \Upsilon_b\left(\frac{Q}{2}\right)\Upsilon_{ib}\left(ib-\frac{iQ}{2}\right) \\
&\qquad = b^{-Qb}\ \gamma\left(\frac{Qb}{2}\right)e^{\pi i[\hat{Q}^2+(\hat{Q}+Q)^2]/8}e^{-2\pi i\tau/4}\frac{\vartheta_1'(0|\tau)\vartheta_1(\frac{Q}{2b}|\tau)}{\vartheta_3(0|\tau)^2}.
\end{aligned}$$

Combining all the powers of e in (6.35), we get

$$\begin{aligned}
-\frac{\pi i}{2}\hat{Q}(Q-\hat{a}) + \frac{\pi i}{8}(-\hat{Q}^2+4\hat{a}Q-4b^2) + \frac{\pi i}{8}(\hat{Q}^2+4/b^2)& \\
= \frac{\pi i}{8}\{-4\hat{Q}Q+4\hat{a}(\hat{Q}+Q)+4(b^{-2}-b^2)\}& \\
= \pi i\hat{a}/b.&
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\frac{C_{\mathrm{sl.}}^{\mathrm{NS}}(a_3,a_2,a_1|b)}{C_{\mathrm{tl.}}^{\mathrm{NS}}(ia_3,ia_2,ia_1|ib)} &= \frac{A_{\mathrm{sl.}}^{\mathrm{NS}}}{A_{\mathrm{tl.}}^{\mathrm{NS}}}b^{-Qb}\ \gamma\left(\frac{Qb}{2}\right)\ e^{\pi i\hat{a}/b} \\
&\times \frac{\vartheta_1'(0|\tau)\vartheta_1(\frac{Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}}{2b}|\tau)\vartheta_1(\frac{\hat{a}-Q}{2b}|\tau)}\prod_{j=1}^{3}\frac{\vartheta_1(\frac{a_j}{b}|\tau)\vartheta_1(\frac{2a_j+Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)\vartheta_1(\frac{\hat{a}_j+Q}{2b}|\tau)}
\end{aligned} \tag{6.40}$$

(e) Analyse T^{NS}: Now we will apply these arguments to the supersymmetric case and see if the restrictions on b persist in this theory. The supersymmetric ratio T^{NS} may be read off

from (6.40) (again up to the scale factor ratio $A^{\rm NS}_{\rm sl.}/A^{\rm NS}_{\rm tl.}$) to be

$$T^{\rm NS}(a_3,a_2,a_1|b) = b^{-b^2}\,\gamma\left(\frac{1+b^2}{2}\right)\,e^{\pi i\hat{a}/b}$$
$$\times\,\frac{\vartheta_1'(0|\tau)\vartheta_1(\frac{Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}}{2b}|\tau)\vartheta_1(\frac{\hat{a}-Q}{2b}|\tau)}\prod_{j=1}^{3}\frac{\vartheta_1(\frac{a_j}{b}|\tau)\vartheta_1(\frac{2a_j+Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)\vartheta_1(\frac{\hat{a}_j+Q}{2b}|\tau)}, \tag{6.41}$$

where we have dropped a factor of b^{-1} as per McElgin, but this may need to be further justified – something we have not yet looked at. So overall, apart from gaining extra ϑ_1's, directly as a result of additional Υ's in the supersymmetric correlators, the coefficient $e^{-\pi i(Q-2\hat{a})/b}$ of the bosonic case now becomes

$$b^{-b^2}\,\gamma\left(\frac{1+b^2}{2}\right) \equiv B(b). \tag{6.42}$$

Now we wish to see if there is a way of obtaining a diagonal two-point function for the $\mathcal{N}=1$ theory may be defined based on the properties of this function $T^{\rm NS}$. behaviour of $T^{\rm NS}$. Using

$$\vartheta_1\Big(z-\frac{Q}{2b}\Big) = -e^{2\pi i z}\vartheta_1\Big(z+\frac{Q}{2b}\Big), \tag{6.43}$$

we can rewrite the original $T^{\rm NS}$ as

$$T^{\rm NS}(a_3,a_2,a_1|b) = -e^{-\pi i\hat{a}/b}\,e^{\pi i\hat{a}/b}\,B(b)\frac{\vartheta_1(\frac{Q}{2b}|\tau)}{\vartheta_1'(0|\tau)}$$
$$\times\left\{\frac{\vartheta_1'(0|\tau)}{\vartheta_1(\frac{\hat{a}}{2b}|\tau)}\prod_{j=1}^{3}\frac{\vartheta_1(\frac{a_j}{b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)}\right\}\times\left\{\frac{\vartheta_1'(0|\tau)}{\vartheta_1(\frac{\hat{a}+Q}{2b}|\tau)}\prod_{j=1}^{3}\frac{\vartheta_1(\frac{2a_j+Q}{2b}|\tau)}{\vartheta_1(\frac{\hat{a}_j+Q}{2b}|\tau)}\right\} \tag{6.44}$$

Since the last two terms cannot be rewritten in the following way

$$\frac{\vartheta_1(\frac{Q}{2b}|\tau)}{\vartheta_1'(0|\tau)}\times\left\{\frac{\vartheta_1'(\hat{x}|\tau)}{\vartheta_1(\hat{x}|\tau)}-\sum_j\frac{\vartheta_1'(\hat{x}_j|\tau)}{\vartheta_1(\hat{x}_j|\tau)}\right\}\times\left\{\frac{\vartheta_1'(\hat{y}|\tau)}{\vartheta_1(\hat{y}|\tau)}-\sum_j\frac{\vartheta_1'(\hat{y}_j|\tau)}{\vartheta_1(\hat{y}_j|\tau)}\right\} \tag{6.45}$$

for $x_j = \frac{1}{2}\big(\frac{a_j}{b}\big)$ and $y_j = \frac{1}{2}\big(\frac{2a_j+Q}{2b}\big)$, which would bring them in a from similar to that of McElgin's crucial identity (6.14), we are unable rewrite (6.41) in terms of a differential equation that is then fed into the sawtooth function $\mathcal{D}_p(x)$ to obtain delta functions. We therefore need to probe the function T^{NS} in its own right to understand its analyticity.

Rewriting T^{NS} in terms of products of ϑ_1 and ϑ_3: Using the standard conventions for Jacobi's theta functions

$$\vartheta_1(z|\tau) \equiv -\vartheta_{11}(z|\tau), \quad \vartheta_3(z|\tau) \equiv \vartheta_{00}(z|\tau) = \vartheta(z|\tau) \tag{6.46}$$

and the identities

$$\vartheta_1(-z|\tau) = -\vartheta_1(z|\tau) \qquad \text{(odd)} \tag{6.47}$$

$$\vartheta_3(-z|\tau) = \vartheta_3(z|\tau) \qquad \text{(even)} \tag{6.48}$$

$$\vartheta_1\Big(z + \frac{Q}{2b}\Big|\tau\Big) = \vartheta_1\Big(z + \frac{1}{2}\tau + \frac{1}{2}\Big|\tau\Big) = e^{-\pi i\tau/4}e^{-\pi i z}\vartheta_3(z|\tau), \tag{6.49}$$

which in particular leads to

$$\vartheta_1\Big(\frac{\hat{a}-Q}{2b}\Big) = -e^{-\pi i\tau/4}e^{\pi i\hat{a}/(2b)}\vartheta_3\Big(\frac{\hat{a}}{2b}\Big|\tau\Big), \tag{6.50}$$

$$\vartheta_1\Big(\frac{Q}{2b}\Big|\tau\Big) = e^{-\pi i\tau/4}\vartheta_3(0|\tau), \tag{6.51}$$

$$\vartheta_1\Big(\frac{2a_j+Q}{2b}\Big|\tau\Big) = e^{-\pi i\tau/4}e^{-\pi i a_j/b}\vartheta_3\Big(\frac{a_j}{b}\Big|\tau\Big), \tag{6.52}$$

$$\vartheta_1\Big(\frac{\hat{a}_j+Q}{2b}\Big) = e^{-\pi i\tau/4}e^{-\pi i\hat{a}_j/(2b)}\vartheta_3\Big(\frac{\hat{a}_j}{2b}\Big|\tau\Big). \tag{6.53}$$

Plugging these into (6.41), we get

$$\begin{aligned} T^{\mathrm{NS}} &= e^{\pi i\hat{a}/b}B(b)\frac{\vartheta_1'(0|\tau)\vartheta_3(0|\tau)}{e^{\pi i\hat{a}/(2b)}\vartheta_1(\frac{\hat{a}}{2b}|\tau)\vartheta_3(\frac{\hat{a}}{2b}|\tau)}\prod_j \frac{e^{-\pi i a_j/b}\vartheta_1(\frac{a_j}{b}|\tau)\vartheta_3(\frac{a_j}{b}|\tau)}{e^{-\pi i\hat{a}_j/(2b)}\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)\vartheta_3(\frac{\hat{a}_j}{2b}|\tau)} \\ &= B(b)\frac{\vartheta_1'(0|\tau)\vartheta_3(0|\tau)}{\vartheta_1(\frac{\hat{a}}{2b}|\tau)\vartheta_3(\frac{\hat{a}}{2b}|\tau)}\prod_j \frac{\vartheta_1(\frac{a_j}{b}|\tau)\vartheta_3(\frac{a_j}{b}|\tau)}{\vartheta_1(\frac{\hat{a}_j}{2b}|\tau)\vartheta_3(\frac{\hat{a}_j}{2b}|\tau)}, \end{aligned} \tag{6.54}$$

where we have used $\sum_i a_i = \sum_i \hat{a}_i = \hat{a}$. This can be rewritten as

$$T^{\mathrm{NS}} = B(b)\frac{\vartheta_3(0|\tau)}{\vartheta_3'(0|\tau)}T^{\mathrm{NS}}_{(\vartheta_1)}T^{\mathrm{NS}}_{(\vartheta_3)} \tag{6.55}$$

where for both $k = 1, 3$, the function has the same form

$$T^{\rm NS}_{(\vartheta_k)} = \frac{\vartheta_k'(0|\tau)}{\vartheta_k(\frac{\hat{a}}{2b}|\tau)} \prod_j \frac{\vartheta_k(\frac{a_j}{b}|\tau)}{\vartheta_k(\frac{\hat{a}_j}{2b}|\tau)}. \tag{6.56}$$

We can see that the term $T^{\rm NS}_{(\vartheta_1)}$ is similar to the one that appears in the bosonic T and for $\hat{x} = \sum_j x_j$, the identity (6.14) can be used to understand that term – in an identical fashion to (6.15). We are therefore left to make sense of the term $T^{\rm NS}_{(\vartheta_3)}$.

A full proof of (6.14) from basic principles is given in the appendix section A.1.2. The task of proving or refuting an identity for ϑ_3 analogous to the one for ϑ_1 (6.14) has proved to be quite challenging. Several attempts based on the properties of Jacobi's theta functions have been made by the author and some of them are documented in the appendix. The ϑ_1 proof in A.1.2 fails to extend to the ϑ_3 case. Moreover, by examining the lowest order terms in q, it can be seen that the ϑ_3 identity cannot hold – the detailed proof is given in A.1.3.

We may however take another approach that makes some progress. Starting once again with the ϑ_1 identity (6.13) that underpins the bosonic results, we differentiate it w.r.t x to get

$$\begin{aligned}
\text{lhs:} \quad & \frac{d}{dx}\Big[\vartheta_1(2x)\prod_j \vartheta_1(2x_j)\Big] \\
& = 2\vartheta_1'(2x)\prod_j \vartheta_1(2x_j) \\
\text{rhs:} \quad & \frac{d}{dx}\Big[\vartheta_1(\hat{x}-x)\prod_j \vartheta_1(\hat{x}_j+x) - \vartheta_1(\hat{x}+x)\prod_j \vartheta_1(\hat{x}_j-x)\Big] \\
& = -\vartheta_1'(\hat{x}-x)\prod_j \vartheta_1(\hat{x}_j+x) + \vartheta_1(\hat{x}-x)\prod_j \vartheta_1(\hat{x}_j+x)\sum_k \frac{\vartheta_1'(\hat{x}_k+x)}{\vartheta_1(\hat{x}_k+x)} \\
& - \vartheta_1'(\hat{x}+x)\prod_j \vartheta_1(\hat{x}_j-x) + \vartheta_1(\hat{x}+x)\prod_j \vartheta_1(\hat{x}_j-x)\sum_k \frac{\vartheta_1'(\hat{x}_k-x)}{\vartheta_1(\hat{x}_k-x)}
\end{aligned}$$

Next we evaluate the derivative at $x = t$, where $t = 1/2 + \tau/2$,

$$\text{lhs:} \qquad 2\vartheta_1'(2t)\prod_j \vartheta_1(2x_j)$$

$$\begin{aligned}\text{rhs:} \qquad & -\vartheta_1'(\hat{x}-t)\prod_j \vartheta_1(\hat{x}_j+t) + \vartheta_1(\hat{x}-t)\prod_j \vartheta_1(\hat{x}_j+t)\sum_k \frac{\vartheta_1'(\hat{x}_k+t)}{\vartheta_1(\hat{x}_k+t)} \\ & -\vartheta_1'(\hat{x}+t)\prod_j \vartheta_1(\hat{x}_j-t) + \vartheta_1(\hat{x}+t)\prod_j \vartheta_1(\hat{x}_j-t)\sum_k \frac{\vartheta_1'(\hat{x}_k-t)}{\vartheta_1(\hat{x}_k-t)}\end{aligned}$$

In order to simplify the rhs, we summarise (and derive) a few ϑ function relations:

$$\vartheta_1(z \pm t) = \pm e^{-\pi i\tau/4 \mp \pi i z}\vartheta_3(z) \tag{6.57}$$

$$\vartheta_1(z - t) = -e^{2\pi i z}\vartheta_1(x + t) \tag{6.58}$$

$$\vartheta_1'(z) = \pi i \vartheta_1(z) - i e^{\pi i\tau/4 + \pi i z}\vartheta_3'(z + t) \tag{6.59}$$

$$\begin{aligned}\vartheta_1'(z + t) &= \pi i e^{-\pi i\tau/4 - \pi i z}\vartheta_3(z) + e^{3\pi i\tau/4 + \pi i z}\vartheta_3'(z + 2t) \\ &= -\pi i e^{-\pi i\tau/4 - \pi i z}\vartheta_3(z) + e^{-\pi i\tau/4 - \pi i z}\vartheta_3'(z) \qquad (6.60) \\ & \qquad \{\because \vartheta_3(z + 2t) = e^{-\pi i\tau - 2\pi i z}\vartheta_3(z), \\ & \qquad \quad \vartheta_3'(z + 2t) = -2\pi i e^{\pi i\tau - 2\pi i z}\vartheta_3(z) + e^{-\pi i\tau - 2\pi i z}\vartheta_3'(z)\}\end{aligned}$$

$$\vartheta_1'(z - t) = -\pi i e^{-\pi i\tau/4 + \pi i z}\vartheta_3(z) - e^{-\pi i\tau/4 + \pi i z}\vartheta_3'(z) \tag{6.61}$$

Hence the rhs becomes,

$$\begin{aligned} & -\{\vartheta_1'(\hat{x}-t) - \vartheta_1'(\hat{x}+t)\}\prod_j \vartheta_1(\hat{x}_j+t) \\ & -e^{2\pi i\hat{x}}\vartheta_1(\hat{x}+t)\prod_j \vartheta_1(\hat{x}_j+t)\Big\{\sum_k \frac{\vartheta_1'(\hat{x}_k+t)}{\vartheta_1(\hat{x}_k+t)}\sum_k \frac{\vartheta_1'(\hat{x}_k-t)}{\vartheta_1(\hat{x}_k-t)}\Big\},\end{aligned}$$

and the full equation takes the following form

$$\begin{aligned}2\vartheta_1'(2t)\prod_j \vartheta_1(2x_j) &= 2e^{-\pi i\tau}\vartheta_3'(\hat{x})\prod_j \vartheta_3(\hat{x}_j) - 2e^{-\pi i\tau}\vartheta_3(\hat{x})\prod_j \vartheta_3(\hat{x}_j)\sum_k \frac{\vartheta_3'(\hat{x}_k)}{\vartheta_3(\hat{x}_k)} \\ &= 2e^{-\pi i\tau}\vartheta_3(\hat{x})\prod_j \vartheta_3(\hat{x}_j)\Big\{\frac{\vartheta_3'(\hat{x})}{\vartheta_3(\hat{x})} - \sum_k \frac{\vartheta_3'(\hat{x}_k)}{\vartheta_3(\hat{x}_k)}\Big\}\end{aligned}$$

or

$$\frac{\vartheta_1'(2t)\prod_j \vartheta_1(2x_j)}{\vartheta_3(\hat{x})\prod_j \vartheta_3(\hat{x}_j)} = e^{-\pi i\tau}\Big\{\frac{\vartheta_3'(\hat{x})}{\vartheta_3(\hat{x})} - \sum_k \frac{\vartheta_3'(\hat{x}_k)}{\vartheta_3(\hat{x}_k)}\Big\} \tag{6.62}$$

Using the Whittaker-Watson notation [82]

$$[r] \equiv \prod_{j=1}^{4} \vartheta_r(x_j) \qquad [\hat{r}] \equiv \prod_{j=1}^{4} \vartheta_r(\hat{x}_j) \tag{6.63}$$

and dividing the identity [82, p.488]

$$[1] - [3] = [\hat{1}] - [\hat{3}], \tag{6.64}$$

by $[\hat{3}]$, we obtain the following result

$$\frac{[3]}{[\hat{3}]} = 1 + \frac{[1]}{[\hat{3}]} - \frac{[\hat{1}]}{[\hat{3}]}. \tag{6.65}$$

The term $[1]/[\hat{3}]$ can be derived from (6.62) and $[\hat{1}]/[\hat{3}] = \prod_{j=1}^{4} \vartheta_1(\hat{x}_j)/\vartheta_3(\hat{x}_j)$. The lhs of this relation $[3]/[\hat{3}]$ is precisely what we need to evaluate $T^{\rm NS}$, which can be written as

$$\begin{aligned}
T^{\rm NS} &= B(b) \times \frac{\vartheta_1'(0)}{\vartheta_1(\hat{x})}\prod_j \frac{\vartheta_1(2x_j)}{\vartheta_1(\hat{x}_j)} \times \frac{\vartheta_3(0)}{\vartheta_3(\hat{x})}\prod_j \frac{\vartheta_3(2x_j)}{\vartheta_3(\hat{x}_j)} \\
&= B(b)\Big\{\frac{\vartheta_1'(0)\prod_j \vartheta_1(2x_j)}{\vartheta_1(\hat{x})\prod_j \vartheta_1(\hat{x}_j)} - \frac{\vartheta_1'(0)\prod_j \vartheta_1(2x_j)}{\vartheta_3(\hat{x})\prod_j \vartheta_3(\hat{x}_j)}\Big\} \\
&= -B(b)\Big[\frac{\vartheta_1'(\hat{x})}{\vartheta_1(\hat{x})} - \sum_j \frac{\vartheta_1'(\hat{x}_j)}{\vartheta_1(\hat{x}_j)} - e^{-\pi i\tau}\frac{\vartheta_1'(0)}{\vartheta_1'(2t)}\Big\{\frac{\vartheta_3'(\hat{x})}{\vartheta_3(\hat{x})} - \sum_j \frac{\vartheta_3'(\hat{x}_j)}{\vartheta_3(\hat{x}_j)}\Big\}\Big] \\
&= -B(b)\Big[\frac{\vartheta_1'(\hat{x})}{\vartheta_1(\hat{x})} - \sum_j \frac{\vartheta_1'(\hat{x}_j)}{\vartheta_1(\hat{x}_j)} - \frac{\vartheta_3'(\hat{x})}{\vartheta_3(\hat{x})} + \sum_j \frac{\vartheta_3'(\hat{x}_j)}{\vartheta_3(\hat{x}_j)}\Big],
\end{aligned} \tag{6.66}$$

since

$$\vartheta_1(0) = 0, \tag{6.67}$$

$$\vartheta_1(z+2t) = e^{-\pi i\tau - 2\pi i z}\vartheta_1(z), \tag{6.68}$$

$$\vartheta_1'(z+2t) = -2\pi i e^{\pi i\tau - 2\pi i z}\vartheta_1(z) + e^{-\pi i\tau - 2\pi i z}\vartheta_1'(z), \tag{6.69}$$

$$\vartheta_1'(2t) = e^{-\pi i\tau}\vartheta_1'(0). \tag{6.70}$$

(f) Understanding unitarity of the Hilbert space:

Having found an expression for T^{NS} in terms of a sum of $\vartheta_k'(z)/\vartheta_k(z)$, we wish to construct a two-point function and determine what representations in the spectrum are allowed. We start with the function just computed (6.66)

$$T^{\mathrm{NS}} = -B(b)\,\big(\Theta_1(x_1,x_2,x_3) - \Theta_3(x_1,x_2,x_3)\big) \tag{6.71a}$$

where

$$\Theta_k(x_1,x_2,x_3) = \frac{\vartheta_k'(\hat{x})}{\vartheta_k(\hat{x})} - \sum_j \frac{\vartheta_k'(\hat{x}_j)}{\vartheta_k(\hat{x}_j)} \tag{6.71b}$$

and $B(b)$ as before (6.42). In the limit that $\epsilon \to 0$ where ϵ is the imaginary part of τ, we have previously seen in the bosonic case that the Θ_1 term can be written as a sum of sawtooth functions. This was due to the fact that each of the quotients ϑ_1'/ϑ_1 in this limit reduced to the sawtooth functions. It is argued in the appendix of [70] that there is an analogous formula for ϑ_3. If we therefore apply McElgin's arguments to the $\mathcal{N}=1$ case, we run into an impossible problem. Essentially, in the limit $\epsilon \to 0$, both ϑ_k'/ϑ_k lead to the same function $\mathcal{D}_1$ for $k=1,3$ and the two terms Θ_1 and Θ_3 in (6.71a) cancel each other. This would lead to $\lim_{\epsilon\to 0} T^{\mathrm{NS}} = 0$ – a situation that would render McElgin's recipe unworkable for the timelike $\mathcal{N}=1$ Liouville theory.

Returning to the α_j-independent term $B(b) = b^{-b^2}\gamma\big(\frac{1+b^2}{2}\big)$ in (6.71a), we note that the function $\gamma(z) = \Gamma(z)/\Gamma(1-z)$ has zeros wherever $\Gamma(1-z)$ has poles, that is, whenever $1-z \in \mathbb{Z}_{\leq 0}$. So we need to look for poles of $\Gamma(\frac{1-b^2}{2})$, or we must have $1-b^2 \in 2\mathbb{Z}_{\leq 0}$. Since the limit we're taking is $\beta = ib \to \sqrt{p/q}$, this leads to demanding

$$\frac{p}{q} = r \in 2\mathbb{Z}_{\leq 0} - 1, \tag{6.72}$$

i.e. $r = -1, -3, -5, \ldots$. In other words, we need p to be an integer multiple of q which is not allowed since (p, q) are coprime. This leads us to conclude that for (p, q) coprime, in the timelike limit where $b \to -i\sqrt{p/q}$, the term $B(b)$ does not produce any zeros. Please note that a few small issues remained to be studied further. These include the b-dependence of the ratio $A^{\mathrm{NS}}_{\mathrm{sl.}}/A^{\mathrm{NS}}_{\mathrm{tl.}}$, which McElgin does not take into account as can be verified by computing T explicitly.

In summary, we have shown that the $\mathcal{N} = 1$ extension of McElgin's arguments does not lead to a satisfactory conclusion. Our future work in this area will include approaches broader than the one taken by McElgin. There are interesting new avenues based on the recent work [79] that may provide further inspiration. Recently, interest has been sparked in a broader class of nonrational theories known as Toda field theories, where not much is yet understood. Liouville can be seen as $sl(2)$ case of the class of $sl(n)$ Toda theories [83]. Understanding their relation to the rational minimal models in a way analogous to [69] for the Liouville case may shed some light on the limits required for analytic continuation.

Chapter 7

Moduli-free String Backgrounds

In this section, we shall bring together all the ingredients discussed above and consider some exact models as examples of string backgrounds. These are algebraic constructions with a total central charge $c = 15$ and do not carry geometrical interpretations. By construction, the internal sector can provide a great deal of choice in selecting models. We shall look at some characteristics that these models share. The closed and open string sectors will be considered separately. A set of numerical programs underpin the results presented here. These programs will not however be discussed in detail; only a summary of some of the main algorithms is given in the appendix B.

In order to have a large-volume interpretation of a Gepner model in terms of a sigma model on a Calabi-Yau manifold [41, 42] without fluxes, it is mandatory that c_{int} is a multiple of 3. For the purposes of having a consistent string background, however, this restriction is dispensable. One can instead start from an external SCFT like the linear dilaton for a suitable ϵ and tensor it with a several $\mathcal{N} = 2$ minimal models giving the correct value of c_{int} without violating any of the constraints imposed by string theory, provided that that total central charge of internal and external sum up to 15,

$$c = c_{\text{ext}} + c_{\text{int}} = 15 \quad \text{with} \quad c_{\text{ext}} = 6 - \epsilon \quad \text{and} \quad c_{\text{int}} = 9 + \epsilon \; . \tag{7.1}$$

In particular, the GSO projection can be performed as usual in a Gepner models, as long as one replaces q_{ext} from (3.81) by the fermion number counting the ψ^{μ} from the linear dilaton theory. The condition that only NS sectors or only R sectors are to be tensorised can, of course, also carried over from the $\epsilon = 0$ case. Modular invariance will be preserved in the same way as in Gepner models, since the projections can still be understood as orbifoldings.

In the examples we have studied, spacetime supersymmetry is broken. Apart from that, the main changes when generalising Gepner's construction to non-zero charge deficit are the loss of a geometric picture involving Calabi-Yau manifolds, and very different spectra of massless and light fields, which will be the topic of the next section.

7.1. Closed String Spectrum

7.1.1. General Remarks on Searching $\epsilon \neq 0$ Models

In this section, we discuss some specific examples, with special focus on the content of massless and light fields. We will find pronounced differences between cases with non-vanishing central charge deficit and Gepner's original construction.

Our internal sector is made up of minimal models of the $\mathcal{N} = 2$ superconformal algebra described previously in section 3.2.1. These models are combined in the spirit of a Gepner model. In a Gepner compactification to $D = 4$ dimensions (with $\epsilon = 0$), the number r of minimal models used satisfies $3 < r \leq 9$ because $1 \leq c(k_i) \leq 3$. The most famous example is $k = (3, 3, 3, 3, 3)$ which (like many other Gepner models) can be related to a sigma model on a Calabi-Yau manifold, in this case the quintic hypersurface in $\mathbb{C}P^4$. Gepner models typically have quite a large number of massless moduli in the NS-sector, coming from the chiral and anti-chiral primaries. These are $\mathcal{N} = 2$ primary fields $|\psi\rangle$ obeying $G^+_{-\frac{1}{2}}|\psi\rangle = 0$ or $G^-_{-\frac{1}{2}}|\psi\rangle = 0$, and it can be shown that they arise from tensor products of minimal model (anti-)chiral fields, labeled $(l_i, \pm l_i, 0)$, such that their conformal dimensions sum up to $\frac{1}{2}$. The Gepner model corresponding to the quintic, e.g., has $101 + 1$ massless moduli of the (chiral,chiral) or (chiral,anti-chiral) type.

Basically, the reason that so many $h_{\text{int}} = \frac{1}{2}$ fields survive the GSO projection is that $c_{\text{int}} = 9$: this forces the levels k_i of the constituent minimal models – or more precisely the numbers $k_i + 2$ – to satisfy conditions on their relative divisibility. The same denominators $k_i + 2$ from the minimal model central charges also occur in the U(1)-charges $q^l_{m,s}$, and due to the tuned divisibility there are many possible choices of (m_i, s_i) satisfying the GSO condition. In view of this connection, one might already expect that our epsilon-Gepner models with $c_{\text{int}} = 9 + \epsilon$ for $\epsilon \neq 0$ will typically have substantially fewer massless modes than Gepner models, or indeed none at all. This expectation is confirmed when studying concrete examples.

Before presenting a few of these, let us make some general observations on tensor products of $\mathcal{N} = 2$ minimal models with $c_{\text{int}} = 9 + \epsilon$. To get $0 < \epsilon \ll 1$, one only needs to consider models with $r = 4, \dots, 8$ tensor factors; for $r = 9$, the smallest possible central charge deficit is $\epsilon = \frac{1}{2}$, for the model with levels $k = (1^8, 2)$.

A small non-zero value of $|\epsilon|$ usually requires at least one of the levels k_i to be rather large, which implies that the i^{th} minimal model contributes a very large number of fields to the tensor product theory; checking the spectrum for light fields is therefore a (somewhat non-trivial) task for the computer.

Having one large k_i also means that there is a narrow mass gap $\Delta m^2 \sim (k_i + 2)^{-1}$ above the lowest-lying states. In [59], this was viewed as a problematic issue, as low mass gaps might be interpreted as a signal for large internal "dimensions" of the compactification. However, the same feature occurs in ordinary Gepner models with $c_{\text{int}} = 9$. There is no reason to perform a limit $\epsilon \to 0$, as is hinted at in [59]; rather, ϵ is a fixed (discrete) input parameter specifying the string background. Still, the narrow mass gaps present the following challenge: For phenomenological applications, one would prefer models with as vast a "desert" above the lowest-mass states as possible, so one is faced with an optimisation problem balancing small ϵ against large k_i.

How small one can make ϵ depends crucially on its sign. For $\epsilon < 0$, one can approach $c_{\text{int}} = 9$ arbitrarily closely from below: e.g., one can start from a tensor product with central charge 6 (from a K3 Gepner model), then tensor with one further $\mathcal{N} = 2$ minimal model, with very large k_i. We will however also present an example manufactured from a $c_{\text{int}} = 9$ Gepner model below.

For $\epsilon > 0$, the situation is very different. When restricting ourselves to tensor products of $\mathcal{N} = 2$ minimal models as outlined above, there is actually a smallest possible positive ϵ that can be achieved. The reason is that the minimal model central charges (3.73) form a discrete series between 1 and 3. More details on series of $c_{\text{int}} > 9$ for our epsilon-Gepner models will be given in the Appendix.

7.1.2. Examples

The $\epsilon > 0$ case:

The "best" $c_{\text{int}} > 9$ model, i.e. the one with the smallest positive ϵ we can construct in this way, has levels $k = (1, 5, 41, 1805)$ and central charge

$$c_{\text{int}} = \frac{4895164}{543907}, \quad \epsilon \approx 1.8 \times 10^{-6}. \tag{7.2}$$

Using the mass formula (5.20) worked out in the previous section, one can compute the spectrum of tachyonic, massless, and "light" fields in the NS and the R sector for this compactification. One finds

- Tachyons: 0 (NS), 0 (R).
- Massless fields: 1 (NS), 0 (R).
- Massive fields with $m^2 < \frac{1}{2}$: 161 (NS), 4 (R).

The single massless state comes from the identity $[(0,0,0), \ldots, (0,0,0)]$ in the internal minimal model and has external oscillator number $N = \frac{1}{2}$, i.e. it gives the graviton. The next lightest field is $[(0,0,0), (0,0,0), (0,0,0), (0,0,0), (2,0,0)]$ and $N = \frac{1}{2}$ and has $m^2 = 2/1807 \approx 10^{-3}$, i.e. its mass sits at about 1000 times the scale set by ϵ. The four massive Ramond-Ramond fields listed here are all rather heavy with $m^2 \sim 1/2$. Just as in the model considered in [59], the spectrum has no spacetime supersymmetry.

A comment on finding such spectra is in order. Since our models typically have one level k_i which is very large, the tensor product of minimal models contains very many fields – about 10^{12} in the above example $k = (1, 5, 41, 1805)$, Sifting through those for suitable h-values to identify tachyonic and massless fields takes too long a time even on a computer. Therefore, one first reduces the number of fields by performing the GSO projection (an easier task since it only involves the m_i and s_i quantum numbers, not the l_i) and then checks the conformal dimensions of the surviving states.

The problem of listing states becomes even more severe for conformal dimensions $h_{\text{int}} > \frac{1}{2}$, as additional $N = 1$ primary fields can arise from linear combinations of descendants of $\mathcal{N} = 2$ primaries. As long as we concentrate on light string modes, we can however ignore those additional states.

We see that our "best" $\epsilon > 0$ model has no massless moduli at all. This may be compared to the closely related model $(1, 5, 41, 1804)$, which has $c_{\text{int}} = 9$ and a geometric interpretation as a Calabi-Yau sigma model. Here, one finds 504 massless moduli, see [39, 40].

The $\epsilon < 0$ case:

From this Calabi-Yau Gepner model, one can of course also obtain a compactification with small negative central charge deficit: The model with $k = (1, 5, 41, 1803)$ has

$$c_{\text{int}} = \frac{4889744}{543305}, \quad \epsilon \approx -1.8 \times 10^{-6}. \tag{7.3}$$

In this case, one finds the following spectrum of low-lying states:

- Tachyons: 0 (NS), 0 (R).
- Massless fields: 1 (NS), 0 (R).
- Massive fields with $m^2 < \frac{1}{2}$: 161 (NS), 0 (R).

Again, the only massless field in the NSNS sector corresponds to the graviton, and the next NS lightest field is still $[(0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (2, 0, 0)]$ with $N = \frac{1}{2}$; its mass is $m^2 = 2/1805 \approx 10^{-3}$.

While the numbers of light fields in the NS and R sector depends crucially on the levels k_i defining the compactification, we can make a few general remarks, exploiting the GSO projection and familiar bounds on conformal dimensions in $\mathcal{N} = 2$ SCFTs, namely $h_{\text{int}} \geq \frac{|q_{\text{int}}|}{2}$ in the NS sector and $h_{\text{int}} \geq \frac{c_{\text{int}}}{24}$ in the R sector. This allows to show that, with the mass formula (5.20), models with $\epsilon > 0$ are tachyon-free and moreover have no massless fields in the Ramond sector. For models with $\epsilon < 0$, one can at least show that there are no tachyonic modes in the NS sector. In case all the $k_i + 2$ are pairwise coprime, one can make some further statements, e.g. concerning absence of R ground states from the GSO-projected theory.

7.2. Open String Spectrum

For $\epsilon < 0$, we find that we need to add a brane to our background in order to keep the dilaton from running away to infinity. This brane should fill the external four-dimensional spacetime, i.e. satisfy Neumann boundary conditions on the external bosons. Branes of that type were already studied in [49] and also in [50]. Boundary states for Gepner models were constructed in [46, 47]; the internal part of these takes the form

$$|\alpha\rangle\!\rangle = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} B_\alpha^{\boldsymbol{\lambda},\boldsymbol{\mu}} |\boldsymbol{\lambda},\boldsymbol{\mu}\rangle\!\rangle \tag{7.4}$$

with

$$B_\alpha^{\boldsymbol{\lambda},\boldsymbol{\mu}} \sim \prod_{j=1}^{r} \frac{\sin \pi \frac{(l_j+1)(L_j+1)}{k_j+2}}{\sin \pi \frac{l_j+1}{k_j+2}}\, e^{i\pi m_j M_j/(k_j+2)}\, e^{-i\pi s_j S_j/2}. \tag{7.5}$$

where the $\boldsymbol{\lambda},\boldsymbol{\mu}$ stand for the collection of (l_i, m_i, s_i) labels, while α is short for the integer (L_i, M_i, S_i) labels of the boundary states themselves. These formulas were written with $\epsilon = 0$ Gepner models in mind, but with the above adaption of the projections, they work just as well in our more general situation. The brane we require for $\epsilon < 0$ can therefore be any (GSO-projected) tensor product of a Neumann brane for the linear dilaton with a boundary state for the internal sector.

Boundary states $|\alpha\rangle\!\rangle_{\text{tot.}}$ in the full theory are a tensor product of boundary states in the external and internal sector, up to a GSO projection. The Neumann boundary conditions in the external sector were studied by [49]. In the original Gepner model boundary state construction [46] where the external sector comprises of four free bosons and four free fermions, only the fermionic modes of the external sector play a role in imposing the GSO conditions. These conditions are implemented by demanding

$$q_{\text{tot}} = q_{\text{int}} + N_{\text{F}} \in 2\mathbb{Z} + 1 \tag{7.6}$$

where q_{tot} is the total U(1)-charge, $q_{\text{int}} = \sum_i q_i$ and N_{F} is the total number of fermionic oscillators in the external theory. Boundary states in $\mathcal{N} = 1$ linear dilaton CFT are a tensor product of boundary states of the bosonic linear dilaton of [49] and free fermions. The procedure therefore is analogous to the case of the free theory.

The open string spectrum for strings with A and B type boundary conditions is described by partition functions given in equations 4.33 and 4.41 of [46].

We also wish to find ϵ-Gepner models with massless Ramond states as this is necessary to have spacetime fermions. For models with massless RR states, there is one further condition we have to keep in mind, namely that the brane we use should not lead to tadpoles for massless RR-fields (tadpoles for massive fields are harmless and are cured by a suitable shift in their vacuum expectation value [21]). Such tadpoles can be cancelled forming suitable (model-dependent) superpositions of the above boundary states (exploiting the fact that the coefficients (7.4) contain roots of unity); cf. the construction of branes carrying torsion charges only for the quintic given in [53]. However, as it turns out, many of the $\epsilon \neq 0$ models we studied so far do not contain any massless RR-fields, so there are no tadpoles to cancel. This will be discussed below.

In the following subsections, we will remark on GSO surviving states in the Ramond sector with the aim to understand generic conditions on labels l, m, s that may lead to reducing the computational workload. We will also discuss some conditions on these labels that may lead to massless states in the sector. We have not been able to numerically find any ϵ-Gepner models with massless RR fields. We nonetheless show a method for cancelling tadpoles should such states exist.

7.2.1. General Remarks on States in the Ramond sector

We consider ϵ-Gepner models with $c_{\text{ext}} = 6 + 6\alpha' V_\mu V^\mu = 6 - \epsilon$, since the charge deficit is $\epsilon = -6\alpha' V_\mu V^\mu$ for a linear dilaton theory. In the R sector,

$$h \geq \frac{c}{24} = \frac{\epsilon}{24} + \frac{3}{8}. \tag{7.7}$$

In particular, equality holds for Ramond ground states $|\psi^l_{m,s}\rangle$, which obey

$$G_0^{\pm}|\psi^l_{m,s}\rangle = (L_0 - \frac{c}{24})|\psi^l_{m,s}\rangle = 0. \tag{7.8}$$

The labels satisfy ranges given in section 3.2.1, in particular we know that $s = \pm 1$ in the Ramond sector and we can see that $q \neq 0$ for a general Ramond state

$$|l, m, \pm 1\rangle, \quad \text{with} \quad h^l_{m,\pm 1} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{1}{8} \quad \text{and} \quad q^l_{m,\pm 1} = \frac{m}{k+2} \mp \frac{1}{2}. \tag{7.9}$$

Comparing with (7.7), we can deduce the general form of a *Ramond ground state* to be

$$|l, \pm(l+1), \pm 1\rangle. \tag{7.10}$$

It is easy to check that $h^l_{l+1,1} = h^l_{-l-1,-1} = \frac{c}{24}$. Since $l = 0, 1, \ldots, k$, there are $2(k+1)$ such states in a single minimal model. Computing the full spectrum of states in a tensor product of minimal models is a computationally intensive task due to the field labels l_i, m_i, s_i taking a large number of values, especially in models where one or more of the minimal models have large levels k_j, the task soon reaches a point of intractability. We therefore look for ways to reduce the number of fields whose masses have to be computed. Since we are only interested in GSO surviving states, we can narrow down our search by first implementing the GSO projection on states in our ϵ-Gepner models. Ramond states of the full internal sector in these models are combinations of ground states of the individual minimal models,

$$\otimes_{i=1}^r |l_i, \pm(l_i+1), \pm 1\rangle \tag{7.11}$$

The GSO projection is implemented by demanding that the total U(1)-charge $q \in 2\mathbb{Z}+1$. We will refer to Ramond primary states other than $|l, \pm(l+1), \pm 1\rangle$ as *non-ground primary* (NGP) states. Unlike the conformal dimension of the above ground states, the U(1)-charge is not independent of the label l,

$$q^l_{l+1,1} = \frac{l+1}{k+2} - \frac{1}{2}, \qquad q^l_{-l-1,-1} = -\frac{l+1}{k+2} + \frac{1}{2}. \tag{7.12}$$

Note that $q^l_{-l-1,-1} = -q^l_{l+1,1}$. Isolating the contributions from $-\sum \frac{s_i}{2}$ term in q, we see that if there are n and n' number of $s_i = \frac{1}{2}$'s and $-\frac{1}{2}$'s respectively with $n + n' = r$, we get

$-\sum \frac{s_i}{2} = \frac{(n'-n)}{2}$, which will be an odd integer whenever $n' - n = 2 \mod 4$. That is, we need

$$\begin{aligned}
&\frac{n'-n}{2} - \sum_{i=1}^{r} \frac{m_i}{k_i+2} \in 2\mathbb{Z}+1. \\
&\frac{(n'-n)\bar{K} + \sum_i m_i \bar{K}_i}{2\bar{K}} = 2p+1, \qquad p \in Z,\ \bar{K} = \mathrm{lcm}(k_i+2) \text{ and } \bar{K}_i = \bar{K}/(k_i+2). \\
&\sum_i m_i \bar{K}_i = 4p\bar{K} + \bar{K}(n - n' + 2).
\end{aligned} \tag{7.13}$$

This last formula provides us with an algorithm to search for R sector GSO survivors. For a given set of r minimal models with levels $(k_1, \ldots, k_r)$, we run over $n = 0, \ldots, r$ and $p = 0, \pm 1, \pm 2, \ldots$ and compute the set of m labels $(m_1, \ldots, m_r)$ that will guarantee fields $\otimes_{i=1}^{n} |l_i, m_i, +1\rangle \otimes_{j=1}^{n'} |l_j, m_j, -1\rangle$ with total $q \in 2\mathbb{Z}+1$. This procedure reduces the possible number of fields whose masses have to be tested, which makes compiling the spectrum computationally less heavy.

7.2.2. Finding Models with Massless Ramond States

We will work with our standard mass formula without the shift. For massless states in the full theory, we are after states that satisfy

$$0 = m^2 = -\Delta_{N,a} = h_{\text{int}} - a + N \tag{7.14}$$

or in other words for $a = 3/8$ in the R-sector, we need to look for states with conformal dimension

$$h_{\text{int}} = \frac{3}{8} - N, \qquad N = 0, 1, \ldots. \tag{7.15}$$

Clearly, there are no massless states at $N > 0$. Expanding h_{int} in terms of conformal dimensions h_i of individual minimal models

$$\begin{aligned}
h_{\text{int}} = \sum_i h_i = \sum_i \frac{l_i(l_i+2) - m_i^2}{4(k_i+2)} - \frac{r}{8} = \frac{3}{8} \\
\sum_i \eta_i \equiv \sum_i \frac{l_i(l_i+2) - m_i^2}{k_i+2} = -f(r),
\end{aligned} \tag{7.16}$$

for $f(r) = (r-3)/2 = \frac{1}{2}, 1, \frac{3}{2}, \ldots, 3$ for $r = 4, 5, \ldots, 9$. (For $c < 9$ models, we may also take $r = 3$ with $f(3) = 0$, but we will not worry about this case as three very large minimal model levels renders the numerical search intractable as explained in the discussion on closed sting sector. Note that $|l_i, l_i, \pm 1\rangle$ is not a standard range Ramond state since $l_i + m_i + s_i$ must be even, despite the fact the other condition $|m_i - s_i| \leq l$ may be met for $s_i = +1$. In general the two conditions, $|m_i - s_i| \leq l_i$ and $l_i + m_i + s_i \in 2\mathbb{Z}$ imply that

- $|l_i, m_i, +1\rangle$ must have $m_i \in \{ \qquad -l_i+1, -l_i+3, \ldots, l_i-3, l_i-1, l_i+1\}$
- $|l_i, m_i, -1\rangle$ must have $m_i \in \{-l_i-1, -l_i+1, -l_i+3, \ldots, l_i-3, l_i-1 \qquad \}$

and $l_i = 0, 1, \ldots, k$. Steps of two are needed since m_i, l_i must have different parities when $s_i = \pm 1$. The two ranges coincide except for the one value in each set, $l_i + 1$ in the first one and $-l_i - 1$ in the second. These precisely give the ground states. So in summary, we have two cases:

$$\text{Ground states:} \qquad |l_i, \pm(l_i+1), \pm 1\rangle. \tag{7.17a}$$

$$\text{NGP states:} \qquad |l_i, m_i, \pm 1\rangle, \quad m_i \in \{-l_i+1, -l_i+3, \ldots l_i-1\}. \tag{7.17b}$$

By looking at these two branches separately, we will see that only combinations of ground and non-ground primary states carries a chance of being massless.

(a) Ramond ground primary states: Let's start by looking at Ramond ground states $|l_i, \pm(l_i+1), \pm 1\rangle$, for whom the summand η_i is independent of l_i, m_i,

$$\eta_i^{(\mathrm{GP})} = -\frac{1}{k_i+2}. \tag{7.18}$$

In other words, in the models where $-\sum_{i=1}^r \eta_i = \sum_{i=1}^r \frac{1}{k_i+2} = f(r)$, the tensor product made up entirely of Ramond ground states will be massless. Unfortunately, models that satisfy this condition necessarily have $c(k_1, \ldots, k_r) = 9$ as may be seen as follows

$$c(k_1, \ldots, k_r) = \sum_{i=1}^r \frac{3k_i}{k_i+2} = \sum_{i=1}^r \frac{3(k_i+2)-6}{k_i+2} = 3r - 6\sum_{i=1}^r \frac{1}{k_i+2}$$

For $c(k_1, \ldots, k_r) = 9$, this reduces to $\sum_{i=1}^r \frac{1}{k_i+2} = f(r)$.

(a) Ramond non-ground primary states: Next we consider the NGP states $|l_i, m_i, \pm 1\rangle$, $m_i \in \{-l_i+1, -l_i+3, \dots l_i-1\}$ in the Ramond sector to see if we can we guess models with such states from the massless condition (7.16). Once again, we collect a few general trivia,

- The state $|0, 0, \pm 1\rangle$ with $\eta_i = 0$ is not in the standard range, so the tensor product must contain states from all minimal models with non-zero contributions η_i.
- States $|l_i, 0, \pm 1\rangle$, $l_i \neq 0$ have $\eta_i > 0$ whereas those of the form $|0, m_i, \pm 1\rangle$, $m_i \neq 0$ have $\eta_i < 0$.
- The only $|0, m_i, \pm 1\rangle$ states are the Ramond ground states $|0, \pm 1, \pm 1\rangle$ which we have covered in the previous section.
- States $|k_i, m_i, \pm 1\rangle$ all have $\eta_i > 0$, so combining them will not satisfy the massless condition.

General NGP states (7.17b) may be written as $|l_i, \pm(l_i - n_i), \pm 1\rangle$, $n_i = 1, 3, 5, \dots, l_i'$ where $l_i' = l_i$ for odd l_i and $l_i' = l_i - 1$ for even l_i. For such states, the η_i turns out to be the same regardless of the sign $+$ or $-$,

$$\eta_i^{(\mathrm{NGP})} = \frac{2(n_i+1)l_i - n_i^2}{k_i+2}. \tag{7.19}$$

These η_i are always positive. Next we try to understand whether for $c \neq 9$ models, we can combine ground states of some models with NGP states of others to produce a massless state in the full tensor product theory. For $r' < r$ number of NGP states and $r - r'$ number of ground states, we get

$$\sum_{i=1}^{r} \eta_i^{(\mathrm{NGP})} = \sum_{i=1}^{r'} \frac{2(n_i+1)l_i - n_i^2 + 1}{k_i+2} - \sum_{i=1}^{r} \frac{1}{k_i+2} \tag{7.20}$$

Once again recall that the second term by itself will not produce the right values of $-f(r)$ for $c \neq 9$ models.

In summary, we have seen above that η_i contributions from all permissible states will be positive unless they are Ramond ground states. A tensor product made up entirely of non-ground primary states therefore will not satisfy $\sum_i \eta_i < 0$, needed for massless states. Furthermore, we have shown above that a tensor product of fields containing only the ground states may only be massless when $c = 9$. Hence, we need to look at the spectrum of those ϵ-Gepner models where the ground states of some models may be tensored with NGP states from other models to form massless states. The numerical search we have conducted so far has not yielded any such models.

7.2.3. Massless Ramond-Ramond Fields and Tadpole Cancellation

As described in the previous section, we have not been able to find an ϵ-Gepner model with massless Ramond-Ramond fields. We have also not been able to rule out that models with such states exist. Presence of massless RR fields would require cancellation of tadpoles. Working with the shifted mass formula of Antoniadis, *et al* [61] however, one does find models with such states – one such model being $(4, 4, 4, 1000)$, for instance. In this section, we work with this model and demonstrate that the Ramond-Ramond charge of boundary states in the open string sector may indeed be cancelled. This is analogous to the torsion branes studied by Brunner and Distler [53], who find a superposition of D-brane boundary states such that the total RR charge vanishes. Superpositions of branes are in general unstable and flow to new boundary states or new branes. One has to make sure that this process does not end up with complete annihilation of the brane. One would like the superposition to be charged. Brunner and Distler [53] find a superposition for the quintic – a $c = 9$ Gepner model – that carries a torsion charge.

The torsion brane setup of [53] uses an explicit construction using a specific $c = 9$ Gepner model, namely the quintic. Here we will apply the same principles to our example $(4, 4, 4, 1000)$. We find a superposition such that the total RR charge of the boundary state represented by the D-brane vanishes. We find a superposition of branes that carries a nontrivial charge under the Greene-Plesser group that makes it stable against any annihilation. We should say from the outset that this construction is model-dependent.

The model $(4, 4, 4, 1000)$ has a central charge of $1502/167 \sim 8.994$ with the charge deficit being $\epsilon = c - 9 = -1/167 \sim -0.005988$. Using the shifted mass formula, the model does indeed have massless RR fields and no tachyons in the spectrum. After taking into account the field identifications, we select states with $m_i > 0$ to work with. In the three $k_j = 4$ $(j = 1, 2, 3)$

models, the Ramond ground states of the individual minimal models have the labels

$$(0,1,1), \qquad (2,3,1), \qquad (3,4,1), \qquad (4,5,1). \tag{7.21}$$

In the $k=1000$ model, we have the following three states

$$(500,501,1), \qquad (667,668,1), \qquad (834,835,1). \tag{7.22}$$

That is $m_j = 1,3,4,5$ for $j=1,2,3$ and $m_4 = 501,668,835$ in the massless Ramond states of the model. The RR charge κ of a boundary state $|\alpha\rangle$ characterised by coefficients (7.4) is given by its projection onto the massless RR states $|\psi\rangle$ [84, 85, 86, 87],

$$\kappa \sim \langle\psi|\alpha\rangle\!\rangle. \tag{7.23}$$

We wish to find a superposition of boundary states $|\alpha\rangle\!\rangle = |\boldsymbol{L},\boldsymbol{M},\boldsymbol{S}\rangle\!\rangle$, such that the charge (7.23) vanishes for all massless RR states in the model. The A and B-type boundary states have coefficients given in (7.4). Let us use the following ansatz for a candidate superposition of branes

$$|B\rangle\!\rangle = \sum_\alpha n_\alpha |\boldsymbol{L},\boldsymbol{M}^{(\alpha)},\boldsymbol{S}\rangle\!\rangle, \tag{7.24}$$

where n_α is the number of times the state α appears in the superposition. Labels $\boldsymbol{L}$ and $\boldsymbol{S}$ take the same values for all branes in the superposition. Any cancellation in (7.23) will therefore come from roots of unity relations involving

$$\sum_\alpha n_\alpha \prod_{j=1}^{r} e^{\pi i \frac{m_j M_j^{(\alpha)}}{k_j+2}} \tag{7.25}$$

only. The rest of the factors in (7.4) can simply be taken out of the α summation.

Listing all massless Ramond states shows that the labels m_i in the spectrum of the model follow a certain pattern: (a) for $m_4 = 668$, there must be two odd and one even m_j labels, $j=1,2,3$, (b) when $m_4 = 501$, all m_j's are odd and (c) for $m_4 = 835$, we see two even and one odd m_j's. By carefully examining various ways of combining the m labels and computing the resulting roots of unity that appear in the RR charge (7.23), we see that the superposition

$$|B_1\rangle\rangle = \sum_{r_i=0,1} |M_1 + 6r_1, M_2 + 6r_2, M_3 + 6r_3, M_4 + 6r_4\rangle\rangle, \tag{7.26}$$

does indeed give a vanishing RR charge. Note that here we have omitted the L and S labels for clarity. The state $|B_1\rangle\rangle$ however is uncharged under the Greene-Plesser group, which for the model $(4, 4, 4, 1000)$ is $G = (\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_{1002})/\mathbb{Z}_{501}$. However we can use the alternative superposition

$$|B_2\rangle\rangle = \sum_{r_i=0,1} |M_1 + 6r_1, M_2 + 6r_2, M_3, M_4 + 6r_4\rangle\rangle, \tag{7.27a}$$

$$|B_3\rangle\rangle = \sum_{r_i=0,1} |M_1 + 6r_1, M_2 + 6r_2, M_3 + 6r_3, M_4\rangle\rangle, \tag{7.27b}$$

which still have vanishing projection onto all massless RR states, but which are charged under the Greene-Plesser group (specifically under the generator shifting M_3 respectively M_4). Due to the non-trivial charge, which plays the same role here as the geometric torsion charge in [53], we can expect that neither $|B_2\rangle\rangle$ nor $|B_3\rangle\rangle$ will undergo complete annihilation under the RG flow to a brane condensate.

Chapter 8

Conclusion

The aim of this thesis has been to use worldsheet conformal field theoretic techniques to study some key problems in string cosmology. The two issues in focus are construction of time-dependent backgrounds and moduli stabilisation and the chapters above treat them largely as independent themes.

The work on the supersymmetric timelike Liouville theory ($\mathcal{N} = 1$) explores a difficult open problem, not just in the supersymmetric case, but also in the bosonic setting. We have been able to show that some of the techniques devised by McElgin [72] that partially work in the bosonic case to produce a unitary Hilbert space in certain special values of the central charge do not extend to the supersymmetric case. As mentioned in the introduction, the recent work of Harlow, Maltz and Witten [79] argues that the restrictions placed by [72] may not be necessary for the validity of the two point function. Future work in this direction may therefore include exploring different ways of defining a two-point function for the analytically continued $\mathcal{N} = 1$ Liouville theory. Since we are also interested in the $\mathcal{N} = 2$ Liouville theory as it has a small central charge suitable for our ϵ-Gepner models, it would be interesting to study its timelike version. The $\mathcal{N} = 2$ Liouville differs considerably from the bosonic and $\mathcal{N} = 1$ Liouville theories and we do not expect this to be a straightforward procedure. Even if the timelike Liouville was perfectly understood, interpretational issues will remain for its suitability to string theoretic cosmological models. The more generic class of conformal field theories known as Toda theories that include Liouville as a special case may provide a rich source of avenues for understanding the issues in Liouville theory. All of this provides ample source of motivation for future work.

As for the application to moduli stabilisation, we have discussed worldsheet CFTs where the split in central charge between the internal and external sector is shifted to $6-\epsilon$ and $9+\epsilon$. This leads to an effective potential for the dilaton which can be stabilised by the addition of suitable D-branes. The result is an effective description, valid for small ϵ that admits a stable AdS_4 vacuum with no massless moduli. We presented explicit compact CFTs in the form of ϵ-Gepner models and also some exact D-brane boundary states. In a sense one might think of these ϵ-Gepner models as CFT analogues of Calabi-Yau compactifications with fluxes. Indeed this raises the possibility that in some cases it might be possible to realise a geometrical limit where the ϵ-Gepner models can be related to Calabi-Yau compactifications with fluxes.

Our models in this sense provide a worldsheet analogue of the flux compactification mechanisms that have been studied in supergravity. The advantages of our construction are that the internal CFT can be described exactly and yet one still obtains a spacetime effective picture that is weakly curved and weakly coupled. On the other hand we have not addressed some important questions. These include combining the techniques used here with models that have a realistic spectrum. In addition one would ultimately like to extend the analysis to include a sort of KKLT mechanism [26] whereby the cosmological constant is lifted to a positive value. Our models also naively break supersymmetry, already because the D-branes do not carry a charge (and hence cannot saturate a lower bound on the mass by the charge that is characteristic of supersymmetric D-branes). However it could be possible with more care to obtain supersymmetric vacua, or at least vacua where supersymmetry is broken at a small scale determined by ϵ.

We also saw that there are in fact a large number of such models. Indeed since there are infinitely many models with $\epsilon < 0$ if we relax the usual Gepner-model constraint that $\epsilon = 0$, it would appear that our models will be rather generic since the separate levels of the minimal models are no longer need to be related by having common factors. In such cases the absence of massless moduli seems to be rather generic (but not universal). We could also use a variety of other CFTs for the internal sector. For example we could consider tensor products of $N = 1$ minimal models, which will provide a "denser set" of ϵ-values to choose from, while still retaining rationality. But other options are available, so there is a chance to find models with a more realistic spectrum. In addition, it would be nice to obtain an exact description of the AdS_4 vacuum at the minimum of the potential. Further work is also required to study lifting the AdS solution to a dS vacuum à la KKLT.

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Appendix A

Miscellaneous Computations

A.1. Some Calculations in Liouville Theory

A.1.1. Theta Function Identities

Properties of Jacobi's theta functions play a central role in computations involving the analytic continuation of Liouville theory. In this section, we recapitulate their definitions and some of the most useful identities .

Definitions:

The original theta function $\vartheta(z|\tau)$ defined as formal series:

$$\boldsymbol{\vartheta(z|\tau)} = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}, \qquad z \in \mathbb{C} \quad \text{and} \quad \operatorname{Im} \tau > 0, \tag{A.1}$$

or

$$\boldsymbol{\vartheta(z|\tau)} = \sum_{n=-\infty}^{\infty} q^{n^2} w^{2n}, \qquad q = e^{\pi i \tau} \text{ (the nome)} \quad \text{and} \quad w = e^{\pi i z}. \tag{A.2}$$

If τ is fixed, this becomes a Fourier series for a periodic entire function of z with period 1; in this case, the theta function satisfies the identity

$$\vartheta(z+1|\tau) = \vartheta(z|\tau). \tag{A.3}$$

Auxiliary theta functions in terms of the original $\vartheta(z|\tau)$:

$$-\vartheta_{11}(z|\tau) = \boldsymbol{\vartheta_1(z|\tau)} \equiv -e^{\pi i\tau/4+\pi i(z+1/2)}\vartheta(z+\tau/2+1/2|\tau) \tag{A.4a}$$
$$\vartheta_{10}(z|\tau) = \boldsymbol{\vartheta_2(z|\tau)} \equiv -e^{\pi i\tau/4+\pi iz}\vartheta\left(z+\tau/2|\tau\right) \tag{A.4b}$$
$$\vartheta_{00}(z|\tau) = \boldsymbol{\vartheta_3(z|\tau)} \equiv \vartheta(z|\tau) \qquad \text{(original)} \tag{A.4c}$$
$$\vartheta_{01}(z|\tau) = \boldsymbol{\vartheta_4(z|\tau)} \equiv \vartheta(z+1/2|\tau) \tag{A.4d}$$

Auxiliary theta functions in terms of q and w:

$$-\vartheta_{11}(z|\tau) = \boldsymbol{\vartheta_1(z|\tau)} = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}w^{2(n+1/2)} \tag{A.5a}$$
$$\vartheta_{10}(z|\tau) = \boldsymbol{\vartheta_2(z|\tau)} = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}w^{2(n+1/2)} \tag{A.5b}$$
$$\vartheta_{00}(z|\tau) = \boldsymbol{\vartheta_3(z|\tau)} = \sum_{n=-\infty}^{\infty} q^{n^2}w^{2n} \tag{A.5c}$$
$$\vartheta_{01}(z|\tau) = \boldsymbol{\vartheta_4(z|\tau)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}w^{2n} \tag{A.5d}$$

A related notation uses $\zeta = w^2$.

The product representation:

$$-\vartheta_{11}(z|\tau) = \boldsymbol{\vartheta_1(z|\tau)} = 2q^{1/4}\sin(\pi z)\prod_{m=1}^{\infty}\left(1-q^{2m}\right)\Big(1-2\cos(2\pi z)q^{2m}+q^{4m}\Big) \tag{A.6a}$$
$$\vartheta_{10}(z|\tau) = \boldsymbol{\vartheta_2(z|\tau)} = 2q^{1/4}\cos(\pi z)\prod_{m=1}^{\infty}\left(1-q^{2m}\right)\Big(1+2\cos(2\pi z)q^{2m}+q^{4m}\Big) \tag{A.6b}$$
$$\vartheta_{00}(z|\tau) = \boldsymbol{\vartheta_3(z|\tau)} = \prod_{m=1}^{\infty}\left(1-q^{2m}\right)\Big(1+2\cos(2\pi z)q^{2m-1}+q^{4m-2}\Big) \tag{A.6c}$$
$$\vartheta_{01}(z|\tau) = \boldsymbol{\vartheta_4(z|\tau)} = \prod_{m=1}^{\infty}\left(1-q^{2m}\right)\Big(1-2\cos(2\pi z)q^{2m-1}+q^{4m-2}\Big) \tag{A.6d}$$

The integral representation:

$$-\vartheta_{11}(z|\tau) = \boldsymbol{\vartheta_1(z|\tau)} = -e^{iz+\pi i\tau/4}\int_{i-\infty}^{i+\infty} \frac{e^{\pi i\tau u^2}\cos(2uz+\pi\tau u)}{\sin(\pi\tau u)}du \tag{A.7a}$$

$$\vartheta_{10}(z|\tau) = \boldsymbol{\vartheta_2(z|\tau)} = -ie^{iz+\pi i\tau/4}\int_{i-\infty}^{i+\infty} \frac{e^{\pi i\tau u^2}\cos(2uz+\pi u+\pi\tau u)}{\sin(\pi\tau u)}du \tag{A.7b}$$

$$\vartheta_{00}(z|\tau) = \boldsymbol{\vartheta_3(z|\tau)} = -i\int_{i-\infty}^{i+\infty} \frac{e^{\pi i\tau u^2}\cos(2uz+\pi u)}{\sin(\pi\tau u)}du \tag{A.7c}$$

$$\vartheta_{01}(z|\tau) = \boldsymbol{\vartheta_4(z|\tau)} = -i\int_{i-\infty}^{i+\infty} \frac{e^{\pi i\tau u^2}\cos(2uz)}{\sin(\pi\tau u)}du \tag{A.7d}$$

Theta functions with 'characteristics':

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) = e^{\pi i a^2\tau+2\pi i a(z+b)}\vartheta(z+a\tau+b|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi(n+a)^2\tau+2\pi i(n+a)(z+b)}. \tag{A.8}$$

$$-\vartheta_{11}(z|\tau) = \boldsymbol{\vartheta_1(z|\tau)} = -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z|\tau) \tag{A.9a}$$

$$\vartheta_{10}(z|\tau) = \boldsymbol{\vartheta_2(z|\tau)} = \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(z|\tau) \tag{A.9b}$$

$$\vartheta_{00}(z|\tau) = \boldsymbol{\vartheta_3(z|\tau)} = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(z|\tau) \tag{A.9c}$$

$$\vartheta_{01}(z|\tau) = \boldsymbol{\vartheta_4(z|\tau)} = \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(z|\tau) \tag{A.9d}$$

Useful Identities:

Fundamental $\vartheta_j(w|\tau) \mapsto \vartheta_k(z|\tau)$ transformations:

(a) $\boldsymbol{z \mapsto -z}$

$$-\vartheta_{11}(-z|\tau) = \boldsymbol{\vartheta_1}(-z|\tau) = -\vartheta_1(z|\tau) \qquad \text{(odd)} \tag{A.10a}$$

$$\vartheta_{10}(-z|\tau) = \boldsymbol{\vartheta_2}(-z|\tau) = \vartheta_2(z|\tau) \qquad \text{(even)} \tag{A.10b}$$

$$\vartheta_{00}(-z|\tau) = \boldsymbol{\vartheta_3}(-z|\tau) = \vartheta_3(z|\tau) \qquad \text{(even)} \tag{A.10c}$$

$$\vartheta_{01}(-z|\tau) = \boldsymbol{\vartheta_4}(-z|\tau) = \vartheta_4(z|\tau) \qquad \text{(even)} \tag{A.10d}$$

(b) $\boldsymbol{z \mapsto z + \frac{1}{2}}$

$$-\vartheta_{11}(z+1/2|\tau) = \boldsymbol{\vartheta_1}(z+1/2|\tau) = \vartheta_2(z|\tau) \tag{A.11a}$$

$$\vartheta_{10}(z+1/2|\tau) = \boldsymbol{\vartheta_2}(z+1/2|\tau) = -\vartheta_1(z|\tau) \tag{A.11b}$$

$$\vartheta_{00}(z+1/2|\tau) = \boldsymbol{\vartheta_3}(z+1/2|\tau) = \vartheta_4(z|\tau) \tag{A.11c}$$

$$\vartheta_{01}(z+1/2|\tau) = \boldsymbol{\vartheta_4}(z+1/2|\tau) = \vartheta_3(z|\tau) \tag{A.11d}$$

(c) $\boldsymbol{z \mapsto z + \frac{1}{2}\tau}$

$$-\vartheta_{11}(z+\tau/2|\tau) = \boldsymbol{\vartheta_1}(z+\tau/2|\tau) = ie^{-\pi i\tau/4-\pi iz}\vartheta_4(z|\tau) \tag{A.12a}$$
$$\vartheta_{10}(z+\tau/2|\tau) = \boldsymbol{\vartheta_2}(z+\tau/2|\tau) = e^{-\pi i\tau/4-\pi iz}\vartheta_3(z|\tau) \tag{A.12b}$$
$$\vartheta_{00}(z+\tau/2|\tau) = \boldsymbol{\vartheta_3}(z+\tau/2|\tau) = e^{-\pi i\tau/4-\pi iz}\vartheta_2(z|\tau) \tag{A.12c}$$
$$\vartheta_{01}(z+\tau/2|\tau) = \boldsymbol{\vartheta_4}(z+\tau/2|\tau) = ie^{-\pi i\tau/4-\pi iz}\vartheta_1(z|\tau) \tag{A.12d}$$

(d) $\boldsymbol{z \mapsto z + \frac{1}{2}\tau + \frac{1}{2}}$

$$-\vartheta_{11}(z+\tau/2+1/2|\tau) = \boldsymbol{\vartheta_1}(z+\tau/2+1/2|\tau) = e^{-\pi i\tau/4-\pi iz}\vartheta_3(z|\tau) \tag{A.13a}$$
$$\vartheta_{10}(z+\tau/2+1/2|\tau) = \boldsymbol{\vartheta_2}(z+\tau/2+1/2|\tau) = -ie^{-\pi i\tau/4-\pi iz}\vartheta_4(z|\tau) \tag{A.13b}$$
$$\vartheta_{00}(z+\tau/2+1/2|\tau) = \boldsymbol{\vartheta_3}(z+\tau/2+1/2|\tau) = ie^{-\pi i\tau/4-\pi iz}\vartheta_1(z|\tau) \tag{A.13c}$$
$$\vartheta_{01}(z+\tau/2+1/2|\tau) = \boldsymbol{\vartheta_4}(z+\tau/2+1/2|\tau) = e^{-\pi i\tau/4-\pi iz}\vartheta_2(z|\tau) \tag{A.13d}$$

Zeros:

$$-\vartheta_{11}(z=0|\tau) = \boldsymbol{\vartheta_1(z=0|\tau)} = 0 \tag{A.14a}$$
$$\vartheta_{10}(z=1/2|\tau) = \boldsymbol{\vartheta_2(z=1/2|\tau)} = 0 \tag{A.14b}$$
$$\vartheta_{00}(z=\tau/2+1/2|\tau) = \boldsymbol{\vartheta_3(z=\tau/2+1/2|\tau)} = 0 \tag{A.14c}$$
$$\vartheta_{01}(z=\tau/2|\tau) = \boldsymbol{\vartheta_4(z=\tau/2|\tau)} = 0 \tag{A.14d}$$

General periodicity properties of the quasi-period τ: For $a, b \in \mathbb{Z}$,

$$\vartheta_{00}(z+a\tau+b|\tau) = \boldsymbol{\vartheta_3(z+a\tau+b|\tau)} = e^{-\pi ia^2\tau-2\pi iaz}\vartheta_3(z|\tau) = q^{-a^2}w^{-2a}\vartheta_3(z|\tau) \tag{A.15}$$

Modular transformations under $\tau \to \tau+1$ and $\tau \to -1/\tau$:

$$-\vartheta_{11}(z|\tau+1) = \boldsymbol{\vartheta_1}(z|\tau+1) = e^{\pi i/4}\vartheta_1(z|\tau) \tag{A.16a}$$
$$\vartheta_{10}(z|\tau+1) = \boldsymbol{\vartheta_2}(z|\tau+1) = e^{\pi i/4}\vartheta_2(z|\tau) \tag{A.16b}$$
$$\vartheta_{00}(z|\tau+1) = \boldsymbol{\vartheta_3}(z|\tau+1) = \vartheta_4(z|\tau) \tag{A.16c}$$
$$\vartheta_{01}(z|\tau+1) = \boldsymbol{\vartheta_4}(z|\tau+1) = \vartheta_3(z|\tau) \tag{A.16d}$$

$$-\vartheta_{11}(z/\tau|-1/\tau) = \boldsymbol{\vartheta_1}(z/\tau|-1/\tau) = -\sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_1(z|\tau) \tag{A.17a}$$
$$\vartheta_{10}(z/\tau|-1/\tau) = \boldsymbol{\vartheta_2}(z/\tau|-1/\tau) = \sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_4(z|\tau) \tag{A.17b}$$
$$\vartheta_{00}(z/\tau|-1/\tau) = \boldsymbol{\vartheta_3}(z/\tau|-1/\tau) = \sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_3(z|\tau) \tag{A.17c}$$
$$\vartheta_{01}(z/\tau|-1/\tau) = \boldsymbol{\vartheta_4}(z/\tau|-1/\tau) = \sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_2(z|\tau) \tag{A.17d}$$

A.1.2. The Crucial ϑ_1 Identity in the Bosonic Timelike Liouville Theory

Here we present a complete proof that makes use of Weierstrauss theta formulae and is based on notation borrowed from Krazer [81]. We start with functions given in equation XLVII of [81],

$$\begin{aligned}
\xi_{\epsilon\epsilon'}^{\rho\rho',\sigma\sigma'}(t,u,v,w) = (-1)^{(\rho+\sigma)\epsilon'+\epsilon+\epsilon'}\vartheta\begin{bmatrix}\epsilon+\rho+\sigma\\ \epsilon'+\rho'+\sigma'\end{bmatrix}(t+u)\ \vartheta\begin{bmatrix}\epsilon+\rho\\ \epsilon'+\rho'\end{bmatrix}(t-u)\\
\vartheta\begin{bmatrix}\epsilon+\sigma\\ \epsilon'+\sigma'\end{bmatrix}(v+w)\ \vartheta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v-w) \qquad \text{(A.18)}\\
\eta_{\epsilon\epsilon'}^{\rho\rho',\sigma\sigma'}(t,u,v,w) = (-1)^{(\rho+\sigma)\epsilon'+\epsilon+\epsilon'}\vartheta\begin{bmatrix}\epsilon+\rho+\sigma\\ \epsilon'+\rho'+\sigma'\end{bmatrix}(t+v)\ \vartheta\begin{bmatrix}\epsilon+\rho\\ \epsilon'+\rho'\end{bmatrix}(t-v)\\
\vartheta\begin{bmatrix}\epsilon+\sigma\\ \epsilon'+\sigma'\end{bmatrix}(w+u)\ \vartheta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(w-u) \qquad \text{(A.19)}\\
\zeta_{\epsilon\epsilon'}^{\rho\rho',\sigma\sigma'}(t,u,v,w) = (-1)^{(\rho+\sigma)\epsilon'+\epsilon+\epsilon'}\vartheta\begin{bmatrix}\epsilon+\rho+\sigma\\ \epsilon'+\rho'+\sigma'\end{bmatrix}(t+w)\ \vartheta\begin{bmatrix}\epsilon+\rho\\ \epsilon'+\rho'\end{bmatrix}(t-w)\\
\vartheta\begin{bmatrix}\epsilon+\sigma\\ \epsilon'+\sigma'\end{bmatrix}(u+v)\ \vartheta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(u-v) \qquad \text{(A.20)}
\end{aligned}$$

and one of many relations, known as the *Weierstrauss theta formulae*, these obey is,

$$\xi_{11}^{\rho\rho',\sigma\sigma'} + \eta_{11}^{\rho\rho',\sigma\sigma'} + \zeta_{11}^{\rho\rho',\sigma\sigma'} = 0. \qquad \text{(A.21)}$$

Here

$$\vartheta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\dots) \equiv \vartheta_2\begin{bmatrix}\epsilon/2\\ \epsilon'/2\end{bmatrix}(\dots) \qquad \text{(A.22)}$$

is defined via

$$\vartheta_r\begin{bmatrix}g\\ h\end{bmatrix}(u) = e^{ag^2+2g(u+\pi ih)}\vartheta(u+ga+\pi ih), \qquad \vartheta(u) = \sum_{n\in\mathbb{Z}} e^{am^2+2mu}, \qquad \text{(A.23)}$$

The parameters g, h are known as the *characteristics* of the ϑ functions and Krazer focuses on the case where $g, h \in \mathbb{Q}$ such that $g = a/r, h = b/r$ for some $a, b \in \mathbb{Z}$ and a common $r \in \mathbb{Z}$. He further restricts to $r = 2$, which is the case we are interested in, and drops the subscript 2 for this special case. This implies that $\rho, \rho', \sigma, \sigma' \in \{0, 1\}$. In order to reproduce (6.13), we first note that ϑ_1 and ϑ_{11} are related to these new ϑ's via

$$\vartheta_1 = -\vartheta_{11} = -\vartheta\begin{bmatrix}1\\1\end{bmatrix}. \tag{A.24}$$

If we take

$$\begin{bmatrix}\rho\\\rho'\end{bmatrix} = \begin{bmatrix}\sigma\\\sigma'\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \tag{A.25}$$

then ξ, η, ζ become

$$\xi_{11}^{00,00} = -\vartheta_1(t+u)\vartheta_1(t-u)\vartheta_1(v+w)\vartheta_1(v-w), \tag{A.26}$$

$$\eta_{11}^{00,00} = -\vartheta_1(t+v)\vartheta_1(t-v)\vartheta_1(w+u)\vartheta_1(w-u), \tag{A.27}$$

$$\zeta_{11}^{00,00} = -\vartheta_1(t+w)\vartheta_1(t-w)\vartheta_1(u+v)\vartheta_1(u-v). \tag{A.28}$$

Next if we let

$$t = x + x_1 \qquad u = x - x_1 \qquad v = x_2 + x_3 \qquad w = x_2 - x_3, \tag{A.29}$$

then using the following three sets of parameters,

$$
\begin{aligned}
1) \qquad t + u &= 2x \\
t - u &= 2x_1 \\
v + w &= 2x_2 \\
v - w &= 2x_3
\end{aligned}
$$

$$
\begin{aligned}
2) \qquad t + w &= x_1 + x_2 + x_3 + x \\
t - w &= x_1 - x_2 + x_3 + x \\
u + v &= -x_1 + x_2 + x_3 + x \\
-(u - v) &= x_1 + x_2 + x_3 - x
\end{aligned}
$$

$$
\begin{aligned}
3) \qquad t + v &= x_1 + x_2 + x_3 + x \\
-(t - v) &= -x_1 + x_2 + x_3 - x \\
-(w + u) &= x_1 - x_2 + x_3 - x \\
w - u &= x_1 + x_2 - x_3 - x
\end{aligned} \tag{A.30}
$$

and the fact that $\vartheta_1(-z) = -\vartheta_1(z)$, McElgin's equation 134 given in (6.13) follows directly from (A.21) when $\epsilon = \epsilon' = 1$ and $\rho = \rho' = \sigma = \sigma' = 0$. This is the identity that allows McElgin to write the product of ϑ_1 and their derivatives in the ratio (6.15) as their sum in the continuation of the bosonic timelike Liouville.

A.1.3. The ϑ_3 Identity for the $\mathcal{N} = 1$ Timelike Liouville Theory

In the $\mathcal{N} = 1$ supersymmetric timelike analytic continuation, the ϑ_3 analogue of the identity proven in the last section is needed. In this section, we will show that this does not hold.

First, note that the ϑ_3 case cannot be proven using the the Weierstrauss theta relations.

To see this, note that the rest of the Weierstrass theta relations are given by

$$\xi_{11} - \eta_{\alpha\alpha'} - \zeta_{\beta\beta'} = 0 \tag{A.31}$$

$$\eta_{11} - \zeta_{\alpha\alpha'} + \xi_{\alpha\alpha'} = 0 \tag{A.32}$$

$$\zeta_{11} - \xi_{\alpha\alpha'} + \eta_{\alpha\alpha'} = 0 \tag{A.33}$$

$$\xi_{\alpha\alpha'} + \eta_{\beta\beta'} + \zeta_{\gamma\gamma'} = 0 \tag{A.34}$$

where $\alpha\alpha', \beta\beta', \gamma\gamma' \in \{00, 10, 01\}$ and we have suppressed the labels $\rho, \rho', \sigma, \sigma'$.

We would like to obtain an analogue of (6.13) for $\vartheta_3 \equiv \vartheta_{00}$ by trying various combinations of $\rho, \rho', \sigma, \sigma'$ in these relations. Clearly, there is no way to use relations (A.31), (A.32) and (A.33) for this purpose as due to the first term in each of them (ξ_{11}, η_{11} and ζ_{11} respectively) there will always be one ϑ_1 in one of the summands. This is because each summand in $\xi_{\epsilon\epsilon'}, \eta_{\epsilon\epsilon'}, \zeta_{\epsilon\epsilon'}$ has at least one ϑ of the kind

$$\vartheta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}. \tag{A.35}$$

Our only hope therefore is (A.34), namely $\xi_{\alpha\alpha'} + \eta_{\beta\beta'} + \zeta_{\gamma\gamma'} = 0$. Indeed we must choose $\epsilon = \epsilon' = \rho = \rho' = \sigma = \sigma' = 0$ or else we will have at least one ϑ_1 with one of the characteristics equal to 1. But this is not allowed according to our understanding of the text that the indices in (A.34) must form a permutation of $(00, 01, 10)$.

This prompts us to try a different approach. We examine the lowest order terms in $q = e^{\pi i \tau}$

in the following definitions of ϑ_k and their derivatives $\vartheta'_k = \partial_z \vartheta_k$ for $k = 1, 3$

$$\begin{aligned}
\vartheta_3(z|\tau) &\equiv \vartheta(z|\tau) \\
&= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}, && \text{(A.36a)} \\
\vartheta'_3(z|\tau) &= 2\pi i \sum_{n=-\infty}^{\infty} n q^{n^2} w^{2n} && \text{(A.36b)} \\
\vartheta_1(z|\tau) &\equiv -e^{\pi i \tau/4 + \pi i (z+1/2)} \vartheta(z + \tau/2 + 1/2|\tau) \\
&= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} w^{2n+1} && \text{(A.36c)} \\
\vartheta'_1(z|\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n (\pi i (2n+1)) q^{(n+1/2)^2} w^{2n+1}, && \text{(A.36d)}
\end{aligned}$$

where $z \in \mathbb{C}$ and $\text{Im } \tau > 0$. Our strategy would be to expand the identity as a power series in q and compare the lowest order terms. The expansions of the two theta functions ϑ_1, ϑ_3 and their derivatives are given by

$$\begin{aligned}
\vartheta_3(z|\tau) &= q^0 w^0 + q(w^{-2} + w^2) + q^4(w^{-4} + w^4) + \mathcal{O}(q^9) \\
&= 1 + q \cos 2\pi z + 2q^4 \cos 4\pi z + \mathcal{O}(q^9) && \text{(A.37a)} \\
\vartheta'_3(z|\tau) &= 2\pi i q(w^2 - w^{-2}) + 4\pi i q^4 (w^4 - w^{-4}) + \mathcal{O}(q^9) && \text{(A.37b)} \\
\vartheta_1(z|\tau) &= q^{1/4}(-i(w - w^{-1})) + q^{9/4}(-i(-w^3 + w^{-3})) + \mathcal{O}(q^{25/4}) && \text{(A.37c)} \\
\vartheta'_1(z|\tau) &= q^{1/4}(\pi(w + w^{-1})) + q^{9/4}(-3\pi(w^3 + w^{-3})) + \mathcal{O}(q^{25/4}). && \text{(A.37d)}
\end{aligned}$$

Before we look at the expansion of the identity we need to prove for $k = 3$,

$$-\frac{\vartheta'_k(0|\tau)}{\vartheta_k(\hat{x}|\tau)} \prod_j \frac{\vartheta_k(2x_j|\tau)}{\vartheta_k(\hat{x}_j|\tau)} = \frac{\vartheta'_k(\hat{x}|\tau)}{\vartheta_k(\hat{x}|\tau)} - \sum_j \frac{\vartheta'_k(\hat{x}_j|\tau)}{\vartheta_k(\hat{x}_j|\tau)}, \tag{A.38}$$

where $x = \sum_j x_j$ and $\hat{x}_j = \hat{x} - 2x_j$, we re-write it in terms only of products and not product and quotients,

$$\vartheta'_k(0|\tau) \prod_j \vartheta_k(2x_j|\tau) + \vartheta'_k(\hat{x}|\tau) \prod_j \vartheta_k(\hat{x}_j|\tau) - \vartheta_k(\hat{x}|\tau) \sum_{j=1}^{3} \vartheta'_k(\hat{x}_j|\tau) \prod_{i \neq j} \vartheta_k(\hat{x}_j|\tau) = 0. \tag{A.39}$$

Not that we also need expressions for $\vartheta'_k(0|\tau)$,

$$\vartheta'_1(0|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n (i\pi(2n+1)) q^{(n+1/2)^2} = 2\pi q^{1/4} - 6\pi q^{9/4} + \mathcal{O}(q^{25/4}) \tag{A.40a}$$

$$\vartheta'_3(0|\tau) = 2\pi i \sum_{n=-\infty}^{\infty} n q^{n^2} = 0 \tag{A.40b}$$

Since $\vartheta'_3(0|\tau) = 0$, the first term drops out from the identity. The lowest order terms of $\vartheta_3(z|\tau)$ is $q^0 = 1$ and of $\vartheta'_3(z|\tau)$ is $2\pi i q(w^2 - w^{-2})$. So we want the expression

$$q(2\pi i(\hat{w}^2 - \hat{w}^{-2})) - 2\pi i q \sum_{j=1}^{3} (\hat{w}_j^2 - \hat{w}_j^{-2}) \qquad \hat{w}_j = e^{\hat{x}_j} \text{ and } \hat{w} = \hat{w}_1 \hat{w}_2 \hat{w}_3. \tag{A.41}$$

to vanish. This can be rewritten as

$$\hat{w}^2 - \hat{w}^{-2} - \sum_{j=1}^{3} (\hat{w}_j^2 - \hat{w}_j^{-2}). \tag{A.42}$$

Taking a special case of $\hat{x}_3 = 0$ or $\hat{w}_3 = 1$, we can see that the expression does not vanish. Therefore the ϑ_3 identity does not hold for all x. A much longer version of this computation leads to the verification of the ϑ_1 version.

A.2. Quantum Fluctuations around Linear Dilaton Background

We start with an effective action of the massless closed string fields ϕ, $g_{\mu\nu}$ and $b_{\mu\nu}$ in the string frame,

$$S^0_{\text{eff}} = \frac{1}{\alpha'} \int d^4X \sqrt{-g}\, e^{-2\phi} \left(R + 4(\mathcal{D}\phi)^2 - \frac{1}{12} H^2 + q\frac{\epsilon}{\alpha'} \right) + \dots \tag{A.43a}$$

and add terms for the 'light' moduli in the NS and R sectors

$$S^{\text{NS}}_{\text{eff}} = \frac{1}{\alpha'} \sum_{\chi_{\text{NS}}} \int d^4X \sqrt{-g}\, e^{-2\phi} \left(-\frac{1}{2}(\partial\chi_{\text{NS}})^2 - \frac{1}{2} m^2_{\text{NS}} \chi^2_{\text{NS}} \right), \tag{A.43b}$$

$$S^{\text{R}}_{\text{eff}} = \frac{1}{\alpha'} \sum_{\chi_{\text{R}}} \int d^4X \sqrt{-g} \left(-\frac{1}{2}(\partial\chi_{\text{R}})^2 - \frac{1}{2} m^2_{\text{R}} \chi^2_{\text{R}} \right), \tag{A.43c}$$

to obtain the full effective action,

$$S_{\text{eff}} = S^0_{\text{eff}} + S^{\text{NS}}_{\text{eff}} + S^{\text{R}}_{\text{eff}}. \tag{A.44}$$

We take the linear dilaton background in the weak field limit of gravity to be given by,

$$\phi = \phi_0 + V_\mu X^\mu + \delta\phi, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad b_{\mu\nu} = b^0_{\mu\nu} + \delta b_{\mu\nu} \tag{A.45}$$

for some small h, $\delta\phi$ and δb and compute the fluctuations by directly substituting these into the above action. By calculating the central charge of the worldsheet CFT in terms of the background vector V_μ we have seen before (5.8) that $\epsilon = -6\alpha' V^2$. By imposing the $\delta\phi$ equation of motion, we will find q to take the value of 2/3 in agreement with section 4.1. We wish to compute the quantum fluctuations of the full action S_{eff} around this background. We also assume that

$$\chi_{\text{NS,R}} = \chi^0_{\text{NS,R}} + \delta\chi_{\text{NS,R}} \tag{A.46}$$

with $\chi^0_{\text{NS,R}} = 0$ a valid classical solutions for fields $\chi_{\text{NS,R}}$. In the following, $\delta^{(i)} f(y)$ denotes the term in f's expansion that is ith order in y.

First and second order corrections to $g_{\mu\nu}$, ϕ and their functions:

First we expand the factor containing the dilaton $e^{-2\phi}$ to 2nd order in $\delta\phi$

$$e^{-2\phi} = e^{-2\phi_0 - 2V\cdot X}\Big(1 - 2\delta\phi + 2(\delta\phi)^2\Big). \tag{A.47}$$

The perturbations of the inverse of the metric are given by [Gasperini p255],

$$\delta^{(1)} g^{\mu\nu} = -h^{\mu\nu}, \qquad \delta^{(2)} g^{\mu\nu} = h^{\mu\lambda} h_\lambda{}^\nu, \tag{A.48}$$

We also need to compute $\delta\sqrt{-\det g}$ given $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, or in matrix notation $g = \eta + h$.

$$\det g = \det(\eta + h) \tag{A.49a}$$
$$= \det\eta\,\det(1 + \eta^{-1}h) \tag{A.49b}$$
$$= -\det\exp\Big\{\eta^{-1}h - \frac{1}{2}\eta^{-1}h\eta^{-1}h\Big\} \tag{A.49c}$$
$$= -\exp\Big\{\mathrm{tr}(\eta^{-1}h) - \frac{1}{2}\mathrm{tr}(\eta^{-1}h\eta^{-1}h)\Big\} \qquad (\det e^A = e^{\mathrm{tr}(A)}) \tag{A.49d}$$
$$= -\Big\{1 + \mathrm{tr}(\eta^{-1}h) - \frac{1}{2}\mathrm{tr}(\eta^{-1}h\eta^{-1}h) + \frac{1}{2}\big(\mathrm{tr}(\eta^{-1}h)\big)^2 + \mathcal{O}(h^3)\Big\} \tag{A.49e}$$

Note that in (A.49b) the notation is $\eta^{-1}h = \eta^{\mu\nu}h_{\mu\nu} = h_\mu{}^\mu$ and $\eta^{-1}h\eta^{-1}h = h_{\mu\nu}h^{\mu\nu}$, and in (A.49c) since we need to expand the exponential in (A.49d) to 2nd order in h, it needs to be compensated by a term proportional to $(\eta^{-1}h)^2$. Since h is small, we can use $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$ and obtain

$$\sqrt{-\det g} \sim 1 + \frac{1}{2}\mathrm{tr}(h) - \frac{1}{4}\mathrm{tr}(h^2) + \frac{1}{4}\big(\mathrm{tr}(h)\big)^2 - \frac{1}{8}\big(\mathrm{tr}(h)\big)^2 + \mathcal{O}(h^3) \tag{A.50a}$$
$$\sim 1 + \frac{1}{2}\mathrm{tr}(h) - \frac{1}{4}\mathrm{tr}(h^2) + \frac{1}{8}\big(\mathrm{tr}(h)\big)^2 + \mathcal{O}(h^3) \tag{A.50b}$$

Here $h = h_\mu{}^\mu$, $h^2 = h_{\mu\nu}h^{\mu\nu}$. So we have the 1st and 2nd order fluctuations for $\sqrt{-g}$ as follows

$$\delta^{(1)}\sqrt{-g} = 1 + \frac{1}{2}h_\mu{}^\mu, \qquad \delta^{(2)}\sqrt{-g} = -\frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \frac{1}{8}h_\mu{}^\mu h_\nu{}^\nu. \tag{A.51}$$

First and second order contributions to the connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{A.52}$$

under these perturbations are

$$\delta^{(1)}\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) - \frac{1}{2}h^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{A.53a}$$
$$\delta^{(2)}\Gamma^\lambda_{\mu\nu} = -\frac{1}{2}h^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) + \frac{1}{2}h^{\lambda\sigma}h_\sigma{}^\rho(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{A.53b}$$

Since we're expanding around the flat background $g_{\mu\nu} = \eta_{\mu\nu}$, these simplify to

$$\delta^{(1)}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \tag{A.54a}$$

$$\delta^{(2)}\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2}h^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}). \tag{A.54b}$$

To compute fluctuations of the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu} \tag{A.55}$$

we simply collect all terms upto 2nd order in $h_{\mu\nu}$.

$$\begin{aligned}
\delta^{(1)}\partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \delta^{(1)}\partial_\nu\Gamma^{\lambda}_{\lambda\mu} &= \frac{1}{2}\eta^{\lambda\rho}(\partial_\lambda\partial_\mu h_{\nu\rho} + \partial_\lambda\partial_\nu h_{\mu\rho} - \partial_\lambda\partial_\rho h_{\mu\nu}) \\
&\quad - \frac{1}{2}\eta^{\lambda\rho}(\partial_\nu\partial_\mu h_{\lambda\rho} + \partial_\nu\partial_\lambda h_{\mu\rho} - \partial_\nu\partial_\rho h_{\mu\lambda}) \\
&= \frac{1}{2}\eta^{\lambda\rho}(\partial_\lambda\partial_\mu h_{\nu\rho} - \partial_\lambda\partial_\rho h_{\mu\nu} - \partial_\nu\partial_\mu h_{\lambda\rho} + \partial_\nu\partial_\rho h_{\mu\lambda}) \\
&= \frac{1}{2}(\partial_\mu\partial^\lambda h_{\nu\lambda} + \partial_\nu\partial^\lambda h_{\mu\lambda} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h) \qquad (h = h_\mu{}^\mu) \\
&= \delta^{(1)} R_{\mu\nu}
\end{aligned} \tag{A.56}$$

and

$$\begin{aligned}
\delta^{(2)}\partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \delta^{(2)}\partial_\nu\Gamma^{\lambda}_{\lambda\mu} = \frac{1}{2}\Big\{ &- \partial_\lambda h^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \\
&- h^{\lambda\rho}(\partial_\lambda\partial_\mu h_{\nu\rho} + \partial_\lambda\partial_\nu h_{\mu\rho} - \partial_\lambda\partial_\rho h_{\mu\nu}) \\
&+ \partial_\nu h^{\lambda\rho}(\partial_\lambda h_{\mu\rho} + \partial_\mu h_{\lambda\rho} - \partial_\rho h_{\lambda\mu}) \\
&+ h^{\lambda\rho}(\partial_\nu\partial_\lambda h_{\mu\rho} + \partial_\nu\partial_\mu h_{\lambda\rho} - \partial_\nu\partial_\rho h_{\lambda\mu})\Big\}.
\end{aligned} \tag{A.57}$$

In the two $\Gamma\Gamma$ terms in $R_{\mu\nu}$, the only quadratic contributions will come from $\delta^{(1)}\Gamma$,

$$\begin{aligned}
\delta^{(1)}(\Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu}) &= \frac{1}{4}\Big\{\eta^{\lambda\sigma}(\partial_\lambda h_{\rho\sigma} + \partial_\rho h_{\lambda\sigma} - \partial_\sigma h_{\lambda\rho}) \times \eta^{\rho\tau}(\partial_\nu h_{\mu\tau} + \partial_\mu h_{\nu\tau} - \partial_\tau h_{\nu\mu}) \\
&\quad -\eta^{\lambda\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}) \times \eta^{\rho\tau}(\partial_\lambda h_{\mu\tau} + \partial_\mu h_{\lambda\tau} - \partial_\tau h_{\lambda\mu})\Big\} \\
&= \frac{1}{4}\Big\{\partial_\rho h_\lambda{}^\lambda \eta^{\rho\tau}(\partial_\nu h_{\mu\tau} + \partial_\mu h_{\nu\tau} - \partial_\tau h_{\nu\mu}) \qquad (\eta^{\lambda\sigma}\partial_\lambda h_{\rho\sigma} = \eta^{\lambda\sigma}\partial_\sigma h_{\lambda\rho}) \\
&\quad -\partial_\nu h_\rho{}^\lambda \eta^{\rho\tau}(\partial_\lambda h_{\mu\tau} + \partial_\mu h_{\lambda\tau} - \partial_\tau h_{\lambda\mu}) \\
&\quad -\partial_\rho h_\nu{}^\lambda \eta^{\rho\tau}(\partial_\lambda h_{\mu\tau} + \partial_\mu h_{\lambda\tau} - \partial_\tau h_{\lambda\mu}) \\
&\quad +\partial^\lambda h_{\nu\rho}\eta^{\rho\tau}(\partial_\lambda h_{\mu\tau} + \partial_\mu h_{\lambda\tau} - \partial_\tau h_{\lambda\mu})\Big\} \\
&= \frac{1}{4}\Big\{\partial_\rho h_\lambda{}^\lambda(\partial_\nu h_\mu{}^\rho + \partial_\mu h_\nu{}^\rho - \partial^\rho h_{\nu\mu}) \\
&\quad -\partial_\nu h_\rho{}^\lambda(\partial_\lambda h_\mu{}^\rho + \partial_\mu h_\lambda{}^\rho - \partial^\rho h_{\lambda\mu}) \\
&\quad -\partial_\rho h_\nu{}^\lambda(\partial_\lambda h_\mu{}^\rho + \partial_\mu h_\lambda{}^\rho - \partial^\rho h_{\lambda\mu}) \\
&\quad +\partial^\lambda h_{\nu\rho}(\partial_\lambda h_\mu{}^\rho + \partial_\mu h_\lambda{}^\rho - \partial^\rho h_{\lambda\mu})\Big\}
\end{aligned} \tag{A.58}$$

By contracting with $\eta^{\mu\nu}$ to get the fluctuations of the Ricci scalar, we obtain

$$\delta^{(1)}R = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h \tag{A.59}$$

and

$$\begin{aligned}
\delta^{(2)}R &= \eta_{\mu\nu}\{\delta^{(2)}\partial_\lambda\Gamma^\lambda_{\mu\nu} - \delta^{(2)}\partial_\nu\Gamma^\lambda_{\lambda\mu} + \delta^{(1)}(\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\nu\mu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\lambda\mu})\} \\
&= \frac{1}{2}\Big\{ - \partial_\lambda h^{\lambda\rho}(2\partial_\mu h^\mu{}_\rho - \partial_\rho h_\mu{}^\mu) - h^{\lambda\rho}(\partial_\lambda\partial_\mu h^\mu{}_\rho + \partial_\lambda\partial^\mu h_{\mu\rho} - \partial_\lambda\partial_\rho h_\mu{}^\mu) \\
&\qquad + \partial^\mu h^{\lambda\rho}(\underline{\partial_\lambda h_{\mu\rho}} + \underline{\partial_\mu h_{\lambda\rho}} - \underline{\partial_\rho h_{\lambda\mu}}) + h^{\lambda\rho}(\underline{\partial^\mu\partial_\lambda h_{\mu\rho}} + \partial^\mu\partial_\mu h_{\lambda\rho} - \partial^\mu\partial_\rho h_{\lambda\mu})\Big\} \\
&\quad + \frac{1}{4}\Big\{ + \partial_\rho h_\lambda{}^\lambda(2\partial^\mu h_\mu{}^\rho - \partial^\rho h_\mu{}^\mu) - \partial^\mu h_\rho{}^\lambda(\underline{\partial_\lambda h_\mu{}^\rho} + \underline{\partial_\mu h_\lambda{}^\rho} - \underline{\partial^\rho h_{\lambda\mu}}) \\
&\qquad - \partial_\rho h^{\mu\lambda}(\underline{\partial_\lambda h_\mu{}^\rho} + \underline{\partial_\mu h_\lambda{}^\rho} - \underline{\partial^\rho h_{\lambda\mu}}) + \partial^\lambda h^\mu{}_\rho(\partial_\lambda h_\mu{}^\rho + \underline{\partial_\mu h_\lambda{}^\rho} - \underline{\partial^\rho h_{\lambda\mu}})\Big\}
\end{aligned} \tag{A.60}$$

$$\begin{aligned}
&= \frac{1}{2}\Big\{ - \partial_\lambda h^{\lambda\rho}(2\partial_\mu h^\mu{}_\rho - \underline{\partial_\rho h_\mu{}^\mu}) - h^{\lambda\rho}(\underline{\partial_\lambda\partial_\mu h^\mu{}_\rho} - \partial_\lambda\partial_\rho h_\mu{}^\mu) \\
&\qquad + h^{\lambda\rho}(\partial^\mu\partial_\mu h_{\lambda\rho} - \underline{\partial^\mu\partial_\rho h_{\lambda\mu}})\Big\} \\
&\quad + \frac{1}{4}\Big\{ + \partial_\rho h_\lambda{}^\lambda(\underline{2\partial^\mu h_\mu{}^\rho} - \partial^\rho h_\mu{}^\mu) + \partial^\lambda h^\mu{}_\rho\partial_\lambda h_\mu{}^\rho\Big\}
\end{aligned} \tag{A.61}$$

Please note that the underlined terms in the above computation and in the remainder of this section signify those that are being worked on in consecutive steps. Here these add up to give

$$\begin{aligned}
\delta^{(2)}R &= \partial^\mu h_\mu{}^\nu\partial_\nu h - \partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda - h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \\
&\quad + \frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h
\end{aligned} \tag{A.62}$$

where we have also relabelled some indices. Once again, we can choose a gauge that helps reduce the number of terms. The traceless $h_{\mu\nu}$ gauge may not be suitable or physically valid. Consider,

$$\partial^\mu h_{\mu\nu} = 0. \tag{A.63}$$

Now we return to the action and compute the full second order fluctuations.

Fluctuations of S^0_{eff}:

Writing $S_{\text{eff}}[b] = -\frac{1}{12\alpha'}\int d^4X\sqrt{-g}e^{-2\phi}H^2$ and substituting $\delta\sqrt{-g}$, $\delta e^{-2\phi}$ and δR,

$$\delta S^0_{\text{eff}} = \frac{1}{\alpha'}\int d^4X\; e^{-2\phi_0-2V\cdot X}\left(1-2\delta\phi+2(\delta\phi)^2\right)\left(1+\frac{1}{2}h_\rho{}^\rho-\frac{1}{4}h_{\rho\sigma}h^{\rho\sigma}+\frac{1}{8}h_\rho{}^\rho h_\sigma{}^\sigma\right) \tag{A.64a}$$

$$\times\Bigg(\partial^\mu\partial^\nu h_{\mu\nu}-\partial^2 h+\partial^\mu h_\mu{}^\nu\partial_\nu h-\partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda-h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \tag{A.64b}$$

$$+\frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h-\frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu}+\frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}-\frac{1}{4}\partial_\mu h\partial^\mu h \tag{A.64c}$$

$$+4V^2-4V^\mu V^\nu h_{\mu\nu}+4V^\mu V^\nu h_{\mu\lambda}h^\lambda{}_\nu-8V^\mu\partial^\nu\delta\phi h_{\mu\nu} \tag{A.64d}$$

$$+4(\partial\delta\phi)^2+8V_\mu\partial^\mu\delta\phi+q\frac{\epsilon}{\alpha'}\Bigg)+\delta S_{\text{eff}}[b] \tag{A.64e}$$

Next we adopt the notation

$$h := h_\mu{}^\mu, \qquad \{h\}^2 := (h_\mu{}^\mu)^2, \qquad h^2 := h_{\mu\nu}h^{\mu\nu}, \tag{A.65}$$

and throw away terms in the factor $\delta\phi\delta\sqrt{-g}$ that are cubic or higher order in the fluctuations h and $\delta\phi$,

$$\delta S^0_{\text{eff}} = \frac{1}{\alpha'}\int d^4X\; e^{-2\phi_0-2V\cdot X}\left(1+\frac{1}{2}h-2\delta\phi-h\delta\phi+2(\delta\phi)^2-\frac{1}{4}h^2+\frac{1}{8}\{h\}^2\right) \tag{A.66a}$$

$$\times\Bigg(\partial^\mu\partial^\nu h_{\mu\nu}-\partial^2 h+\partial^\mu h_\mu{}^\nu\partial_\nu h-\partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda-h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \tag{A.66b}$$

$$+\frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h-\frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu}+\frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}-\frac{1}{4}\partial_\mu h\partial^\mu h \tag{A.66c}$$

$$+4V^2-4V^\mu V^\nu h_{\mu\nu}+4V^\mu V^\nu h_{\mu\lambda}h^\lambda{}_\nu-8V^\mu\partial^\nu\delta\phi h_{\mu\nu} \tag{A.66d}$$

$$+4(\partial\delta\phi)^2+8V_\mu\partial^\mu\delta\phi+q\frac{\epsilon}{\alpha'}\Bigg)+\delta S_{\text{eff}}[b] \tag{A.66e}$$

Next (a) we expand the brackets and keep terms up to 2nd order in the fluctuation fields $\delta\phi$ and h, (b) note that linear terms $\frac{1}{2}h-2\delta\phi$ only survive when multiplied with $\partial^\mu\partial^\nu h_{\mu\nu}-\partial^2 h+V^2+4V^\mu V^\nu h_{\mu\nu}+8V_\mu\partial^\mu\delta\phi+q\epsilon/\alpha'$, and (c) that the quadratic terms $2(\delta\phi)^2-h\delta\phi-\frac{1}{4}h^2+\frac{1}{8}\{h\}^2$

only survive when multiplied with the constant $8V^2$. This leads to

$$\delta S^0_{\text{eff}} = \frac{1}{\alpha'}\int d^4X\, e^{-2\phi_0-2V\cdot X} \tag{A.67a}$$

$$\times\Bigg\{\Bigg(\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h + \partial^\mu h_\mu{}^\nu\partial_\nu h - \partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda - h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \tag{A.67b}$$

$$+\frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h \tag{A.67c}$$

$$+4V^2 - 4V^\mu V^\nu h_{\mu\nu} + 4V^\mu V^\nu h_{\mu\lambda}h^\lambda{}_\nu - 8V^\mu\partial^\nu\delta\phi h_{\mu\nu} + 4(\partial\delta\phi)^2 + \underline{8V_\mu\partial^\mu\delta\phi} + q\frac{\epsilon}{\alpha'}\Bigg) \tag{A.67d}$$

$$+\Bigg(\frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{2}h\partial^2 h + 2hV^2 - 2hV^\mu V^\nu h_{\mu\nu} + 4hV_\mu\partial^\mu\delta\phi + \frac{1}{2}hq\frac{\epsilon}{\alpha'} \tag{A.67e}$$

$$-2\delta\phi\partial^\mu\partial^\nu h_{\mu\nu} + 2\delta\phi\partial^2 h - \underline{8V^2\delta\phi} + 8\delta\phi V^\mu V^\nu h_{\mu\nu} - 16V_\mu\delta\phi\partial^\mu\delta\phi - \underline{2q\frac{\epsilon}{\alpha'}\delta\phi}\Bigg) \tag{A.67f}$$

$$+\Big(4V^2 + q\frac{\epsilon}{\alpha'}\Big)\Big(2(\delta\phi)^2 - h\delta\phi - \frac{1}{4}h^2 + \frac{1}{8}\{h\}^2\Big)\Bigg\} + \delta S_{\text{eff}}[b]. \tag{A.67g}$$

In order to verify the ϵ condition (5.8), we first look at terms linear in $\delta\phi$ (underlined in the above eqquation) while assuming $h_{\mu\nu} = 0$,

$$e^{-2V\cdot X}(-8V^2\delta\phi - 2q\frac{\epsilon}{\alpha'}\delta\phi + 8V_\mu\partial^\mu\delta\phi) = e^{-2V\cdot X}\Big(8V^2 - 2q\frac{\epsilon}{\alpha'}\Big)\delta\phi \qquad \text{(integrate by parts)}$$

The $\delta\phi$ e.o.m. gives

$$8V^2 = 2q\frac{\epsilon}{\alpha'} \qquad \text{and using (5.8)} \qquad q = -\frac{2}{3} \tag{A.68}$$

which matches the factors in the effective action. Imposing the e.o.m., the $\delta\phi$ terms drop out and we can substitute ϵ and q in the rest of the expression such that the terms proportional toV^2 and ϵ combine to give $8V^2$,

$$\delta S^0_{\text{eff}} = \frac{1}{\alpha'}\int d^4X\ e^{-2\phi_0-2V\cdot X} \tag{A.69a}$$

$$\times\Bigg\{\Bigg(\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h + \partial^\mu h_\mu{}^\nu\partial_\nu h - \partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda - h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \tag{A.69b}$$

$$+\frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h \tag{A.69c}$$

$$+8V^2 + 4(\partial\delta\phi)^2 - 4V^\mu V^\nu h_{\mu\nu} + 4V^\mu V^\nu h_{\mu\lambda}h^\lambda{}_\nu - 8V^\mu\partial^\nu\delta\phi h_{\mu\nu}\Bigg) \tag{A.69d}$$

$$+\Bigg(\frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{2}h\partial^2 h + 4hV^2 - 2hV^\mu V^\nu h_{\mu\nu} + 4hV_\mu\partial^\mu\delta\phi \tag{A.69e}$$

$$-2\delta\phi\partial^\mu\partial^\nu h_{\mu\nu} + 2\delta\phi\partial^2 h + 8\delta\phi V^\mu V^\nu h_{\mu\nu} - 16V_\mu\delta\phi\partial^\mu\delta\phi\Bigg) \tag{A.69f}$$

$$+\Bigg(16V^2(\delta\phi)^2 - 8V^2 h\delta\phi - \frac{1}{2}V^2h^2 + V^2\{h\}^2\Bigg)\Bigg\} + \delta S_{\text{eff}}[b]. \tag{A.69g}$$

Further note that after integration by parts, $e^{-2V\cdot X}\partial^2 h$ leads to $4e^{-2V\cdot X}V^2 h$ which cancels the other such term, and $e^{-2V\cdot X}\partial^\mu\partial^\nu h_{\mu\nu}$ leads to $4e^{-2V\cdot X}V^\mu V^\nu h_{\mu\nu}$, which is also cancelled. We are therefore left with

$$\delta S^0_{\text{eff}} = \frac{1}{\alpha'}\int d^4X\ e^{-2\phi_0-2V\cdot X} \tag{A.70a}$$

$$\times\Bigg(\underline{\partial^\mu h_\mu{}^\nu\partial_\nu h} - \partial_\mu h^{\mu\lambda}\partial_\nu h^\nu{}_\lambda - h^{\mu\nu}\partial_\mu\partial_\lambda h^\lambda{}_\nu \tag{A.70b}$$

$$+\frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial^2 h_{\mu\nu} + \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h \tag{A.70c}$$

$$+8V^2 + 4(\partial\delta\phi)^2 + 4V^\mu V^\nu h_{\mu\lambda}h^\lambda{}_\nu - 8V^\mu\partial^\nu\delta\phi h_{\mu\nu} \tag{A.70d}$$

$$+\underline{\frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu}} - \frac{1}{2}h\partial^2 h - \underline{2hV^\mu V^\nu h_{\mu\nu}} + 4hV_\mu\partial^\mu\delta\phi \tag{A.70e}$$

$$-\underline{2\delta\phi\partial^\mu\partial^\nu h_{\mu\nu}} + 2\delta\phi\partial^2 h + 8\delta\phi V^\mu V^\nu h_{\mu\nu} - \underline{16V_\mu\delta\phi\partial^\mu\delta\phi} \tag{A.70f}$$

$$+\underline{16V^2(\delta\phi)^2} - 8V^2h\delta\phi - \frac{1}{2}V^2h^2 + V^2\{h\}^2\Bigg) + \delta S_{\text{eff}}[b]. \tag{A.70g}$$

Further note that

$$\begin{aligned} e^{-2V\cdot X}\delta\phi\partial^\mu\delta\phi &= (\text{total derivative}) - e^{-2V\cdot X}\delta\phi\partial^\mu\delta\phi + 2e^{-2V\cdot X}V^\mu(\delta\phi)^2 \\ &= e^{-2V\cdot X}V^\mu(\delta\phi)^2 \end{aligned} \tag{A.71}$$

and therefore,

$$16e^{-2V\cdot X}V_\mu \delta\phi \partial^\mu \delta\phi = 16e^{-2V\cdot X}V^2(\delta\phi)^2. \tag{A.72}$$

This cancels the term in (A.70g) with opposite sign and we are left with no mass term for $\delta\phi$. This provides further justification that the mass formula for our linear dilaton CFT indeed does not obtain a mass shift. Further simplifications to the expression may be made, but we will not pursue these calculations any further.

Appendix B

Numerical Results

In section 7.1.2, we looked at two example models $(1, 5, 41, 1803)$ and $(1, 5, 41, 1805)$ for the cases $\epsilon < 0$ and $\epsilon > 0$ respectively. We also noted that they could be obtained by perturbing the largest level k_i of a $c = 9$ ($\epsilon = 0$) Gepner model $(1, 5, 41, 1804)$. A host of techniques were deployed to search for these models, including numerical programs. In this section, we summarise some of these techniques and list a few more examples.

B.1. Summary of the Numerics

It should be noted that the spectrum of models is computed exactly – numerically does not imply 'approximately. Finding models with small $|\epsilon|$ is a numerical problem. For $\epsilon > 0$, there is an algorithm described in section B.1.2 that produces the best possible model, i.e. the first model with central charge over 9. For the case of $\epsilon < 0$, the smallest ϵ would be achieved in the the limiting case of $k_j \to \infty$ for some level k_j in the tensor product. Due to the computational intractability of computing the spectrum of a large k model, we cannot realistically take k to be arbitrarily large.

B.1.1. Searching for Models with Fixed Central Charge C

To achieve a particular value C of the central charge, we can tensor r minimal models with levels $k = (k_1, k_2, \ldots, k_r)$ such that their central charges add up to C,

$$c(k_1, \ldots, k_r) = C, \qquad c(k_1, \ldots, k_r) = \sum_{i=1}^{r} c(k_i) = \sum_{i=1}^{r} \frac{3k_i}{k_i+2}. \tag{B.1}$$

Our strategy is to start with $r = 4$ and try different combinations of k_i, testing for each the central charge $c(k_1, \ldots, k_r)$. Let's look a few obvious issues,

(1) What happens when we hit a k that has $c \geq C$?

We need to decide if to keep looking for models with the current value of r or move on to $r+1$. Given a set of levels with $c \geq C$ and a current index σ, we have two cases to consider

(a) $\boldsymbol{\sigma = 1}$:

- Given that $k_{j+1} \geq k_j$, for all j, we know that increasing k_σ will take us further over C.
- Stop looking for models with r number of minimal models. Move up to $r+1$.

(b) $\boldsymbol{1 < \sigma \leq r}$:

- Stop looking at models with bigger k_σ.
- Increment $k_{\sigma-1}$ and set all $k_j = k_{\sigma-1}$ with $j \geq \sigma$.
- Move on to $\sigma - 1$ as the current index.

(2) What to do when $c < C$?

The obvious thing is to keep increasing k_σ until we reach $c \geq C$. The problem with this is that if $c(k_1, \ldots, k_{\sigma-1}) \leq C - 3(r-\sigma)$, we will never reach C since $c(k_i)$ is bounded from above by 3 as $k_i \to \infty$. We therefore first test if $c(k_1, \ldots, k_{\sigma-1})$ is less than or equal to $C - 3(r-\sigma)$ or not.

- If it is, stop increasing k_σ and move the current index σ to the previous one, i.e., $\sigma \to \sigma - 1$. Also increment $k_{\sigma-1}$ and set all $k_j = k_{\sigma-1}$ with $j \geq \sigma$. Recall that being equal to doesn't help either, as we would need $k_j = \infty$ for all $j \geq \sigma$.

- If on the other hand $c(k_1, \ldots, k_{\sigma-1}) > C - 3(r - \sigma)$, we know that increasing this k_σ has a chance of taking us to $c = C$. So,
  - We keep increasing $k_\sigma \to k_\sigma + 1$ until c reaches C.
  - When we achieve $c \geq C$, we stop exploring this branch further.
  - Increment $k_{\sigma-1}$ and set all $k_j = k_{\sigma-1}$ with $j \geq \sigma$.
  - Move on to $\sigma - 1$ as the current index.

Algorithm in pseudocode: In the following, we construct a step-by-step algorithm.

```
C := 9            // the required central charge
models := []   // a list for storing models found with required central charge
r := 4
while (r <= 9) {
    k := []     // a list of minimal model levels
    Initialise k with each k[i] := 1
    c := charge(k)
    σ := r - 1   // the current index to keep track of the level k[σ] being incremented
    keepGoing := true
    while (keepGoing == true) {
        if (c >= C) {
            if (c == C) {
                models.add(k)
            }
            if (σ == 0) {    // k[0] is the first entry
                keepGoing = false       // stop looking for models at this r
            }
            else {            // i.e., (0 < σ <= r - 1)
                ++k[σ-1]
                for (j = σ; j < r; ++j)   k[j] = k[σ-1]
                --σ
            }
        }
        else {                      // i.e., if (c < C)
            testc := charge(k[1])+...+charge[σ-1]) + 3 * (r - σ)   // test charge
            if (testc <= C ) {
                ++k[σ-1]
                for (j = σ; j < r; ++j)   k[j] = k[σ-1]
                --σ
            }
            else {                    // i.e., (testc > C )
                while (c < C) {     // loop over k[σ] until c >= C
                    ++k[σ]
                    c = charge(k)
                }
```

```
:                   if (c == C) {
:                       models.add(k)
:                   }
:                   ++k[σ-1]
:                   for (j = σ; j < r; ++j)    k[j] = k[σ-1]
:                   --σ
:               }
:           }
:           if (σ == 0) {                // k[0] is the first entry
:               keepGoing = false        // stop looking for models at this r
:           }
:       }     // end of:  while (keepGoing == true)
:   ++r       // move on to next r
:   }
```

B.1.2. The 'Best' $\epsilon > 0$ Model

We start by commenting on how one goes about finding the 'best' $\epsilon > 0$ models, that is, models with smallest $|\epsilon|$. In the following, a tensor product of r minimal models with levels $(k_1, k_2, \ldots, k_r)$ will be written in an ascending order of their levels, i.e., $k_{i-1} \leq k_i$. Since $c(k+1) > c(k)$, $c(k'_1, k'_2, \ldots, k'_r) \leq c(k_1, k_2, \ldots, k_r)$ for all $k'_i \leq k_i$. The algorithm we use finds the first $(k_1, k_2, \ldots, k_r)$, $(k_{i-1} \leq k_i)$ model with central change $9+\epsilon$, $\epsilon > 0$, i.e.,

$$c(k_1, k_2, \ldots, k_r - 1) \leq 9 \qquad \text{and} \qquad c(k_1, k_2, \ldots, k_r) > 9.$$

Each subsequent model $(k_1, k_2, \ldots, k_r+1)$ will have a central charge greater than $9+\epsilon$. Given an upperbound ϵ', the algorithm also determines the model $(k_1, k_2, \ldots, k_{r-1}, k'_r)$ with highest k'_r such that $c(k_1, k_2, \ldots, k'_r) \leq 9 + \epsilon'$.

In the following table, we list models with $0 < \epsilon \leq 0.05$. The total number of models in this range is infinite and we list a few series of models from $(k_1, k_2, \ldots, k_r)$ to $(k_1, k_2, \ldots, k'_r)$ as defined above. We find the total number of these series to be 160.

Note that the model (1,5,87,86) has a smaller ϵ than that of (1,5,87,87) but since we order models in ascending order of their levels, $(1, 5, 87, 86)$ is included in the $(1, 5, 86, 86)$ to $(1, 5, 86, 241)$ series. Also note that there is no $r = 8$ model with $0 < \epsilon \leq 0.05$ as the first model after $c(1^7, 4) = 9$ gives $c(1^7, 5) \sim 9.14$.

Table B.1: **Best $\epsilon > 0$ models**

r	$(k_1, k_2, \ldots, k_r)$	c	ϵ	$(k_1, k_2, \ldots, k'_r)$
4	(1,5,41,1805)	4895164/543907	$1/543907 \sim 1.84 \times 10^{-6}$	$(1, 5, 41, \infty)$
	(1,5,42,923)	1282051/142450	$1/142450 \sim 7 \times 10^{-6}$	$(1, 5, 42, \infty)$
	(1,5,45,393)	1169596/129955	$1/129955 \sim 7.7 \times 10^{-6}$	$(1, 5, 45, \infty)$
	(1,5,43,629)	596296/66255	$1/66255 \sim 1.5 \times 10^{-5}$	$(1, 5, 43, \infty)$
	(1,5,44,482)	350659/38962	$1/38962 \sim 2.5 \times 10^{-5}$	$(1, 5, 44, \infty)$
	(2,3,19,419)	265231/29470	$1/29470 \sim 3.4 \times 10^{-5}$	$(1, 5, 19, \infty)$
	(3,3,9,109)	18316/2035	$1/2035 \sim 4.9 \times 10^{-4}$	(3,3,9,1318)
	(4,4,5,41)	2710/301	$1/301 \sim 3.32 \times 10^{-3}$	(4,4,5,62)
	(1,5,86,86)	1387/154	$1/154 \sim 6.5 \times 10^{-3}$	(1,5,86,241)
	(1,5,87,87)	5612/623	$5/623 \sim 8.03 \times 10^{-3}$	(1,5,87,233)
	(5,5,7,8)	947/105	$2/105 \sim 1.9 \times 10^{-2}$	(5,5,7,8)
5	(1,1,2,11,155)	36739/4082	$1/4082 \sim 2.45 \times 10^{-4}$	$(1, 1, 2, 11, \infty)$
	(1,1,2,15,39)	12547/1394	$1/1394 \sim 7.2 \times 10^{-4}$	(1,1,2,15,59)
	(1,1,2,12,83)	10711/1190	$1/1190 \sim 8.4 \times 10^{-4}$	(1,1,2,12,278)
	(1,1,2,13,59)	5491/610	$1/610 \sim 1.64 \times 10^{-3}$	(1,1,2,13,118)
	(2,2,2,4,11)	235/26	$1/26 \sim 3.85 \times 10^{-2}$	(2,2,2,4,11)
6	(1,1,1,1,5,41)	2710/301	$1/301 \sim 3.32 \times 10^{-3}$	(1,1,1,1,5,62)
	(1,1,1,1,6,23)	901/100	$1/100 = 10^{-2}$	(1,1,1,1,6,28)
	(1,1,1,1,7,17)	514/57	$1/57 \sim 1.75 \times 10^{-2}$	(1,1,1,1,7,19)
7	(1,1,1,1,1,2,11)	235/26	$1/26 \sim 3.85 \times 10^{-2}$	(1,1,1,1,1,2,11)
	(1,1,1,1,1,3,6)	181/20	$1/20 = 5 \times 10^{-2}$	(1,1,1,1,1,3,6)

The first model in the table $(1, 5, 41, 1805)$ has the smallest possible positive $\epsilon \sim 1.84 \times 10^{-6}$. Note that $c(1, 5, 41, 1804) = 9$. The fact that $(1, 5, 41, 1805)$ has the smallest ϵ can be shown by starting with the model $(1^4) = (k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1)$ and incrementing k_i one at a time resulting in the smallest step increase in the central charge. If k_i, $i = 1, 2, 3$ are kept fixed at 1 and k_4 is increased, we will never obtain a total central charge of 9. Hence we find a bound $k_3 > 1$. By induction on this argument, we see that at least 3 k_i's must be greater than 1. We repeat this procedure until the combination $(1, 5, 41, 1804)$ is obtained with central charge 9 and the very next model $(1, 5, 41, 1805)$ provides us with the smallest possible change in the central charge over 9. We can also see that there are infinite number of $r = 4$, $c \leq 9.05$ models with three levels $k_1 = 1, k_2 = 5, k_3 = 41$ by observing that $c(1, 5, 41) < 6.05$ and $c(k)$ is bounded from above by 3.

B.2. Model Listings

Here we summarise data calculated for a few example models. The mass scale for light fields is set to $M^2 = 1000\epsilon$ in all cases below.

Table B.2: **Spectrum of some $\epsilon > 0$ models**

Levels k_i	**Central charge** c	**Charge deficit** ϵ	**Tach-yons**	**Massless fields**	**Light fields** $(m^2 \leq M^2)$
(1,5,41,1805)	4895164/543907	1.84×10^{-6}	0	1 (NS), 0 (R)	1 (NS), 0 (R)
(1,5,41,1806)	2448937/272104	3.68×10^{-6}	0	1 (NS), 0 (R)	2 (NS), 0 (R)
(1,5,42,923)	1282051/142450	7×10^{-6}	0	1 (NS), 0 (R)	2 (NS), 0 (R)
(1,5,45,393)	1169596/129955	7.7×10^{-6}	0	1 (NS), 0 (R)	1 (NS), 0 (R)
(1,5,43,629)	596296/66255	1.5×10^{-5}	0	2 (NS), 0 (R)	4 (NS), 0 (R)
(1,5,44,482)	350659/38962	2.5×10^{-5}	0	2 (NS), 0 (R)	3 (NS), 0 (R)

Table B.3: **Spectrum of some $\epsilon < 0$ models**

Levels k_i	**Central charge** c	**Charge deficit** ϵ	**Tach-yons**	**Massless fields**	**Light fields** $(m^2 \leq M^2)$
(1,5,41,1803)	4889744/543305	-1.84×10^{-6}	0	1 (NS), 0 (R)	1 (NS), 0 (R)
(1,5,41,1802)	2443517/271502	-3.68×10^{-6}	0	1 (NS), 0 (R)	2 (NS), 0 (R)
(1,5,42,921)	1279277/142142	-7.04×10^{-6}	0	1 (NS), 0 (R)	2 (NS), 0 (R)
(1,5,43,627)	594404/66045	-1.51×10^{-5}	0	2 (NS), 0 (R)	4 (NS), 0 (R)
(1,5,44,480)	349208/38801	-2.58×10^{-5}	0	2 (NS), 0 (R)	3 (NS), 0 (R)
(1,5,45,392)	583315/64813	-3.1×10^{-5}	0	1 (NS), 0 (R)	3 (NS), 0 (R)

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